A Mathematical Consideration for the Effect of Regional Lockdown on Endemic Size

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What is consider

Regional lockdown can be considered as a policy to suppress the spread of a transmissible disease when the epidemic breaks out. In this work, we consider a simple mathematical model to theoretically discuss the efficiency of the lockdown, for which we introduce different restrictions on the mobility of individuals and define four types of lockdown: complete lockdown, strong lockdown, weak lockdown type 1, and weak lockdown type 2. The efficiencies of those lockdowns are compared according to the endemic size, that is, the number of infective individuals at the endemic equilibrium.

Assumptions

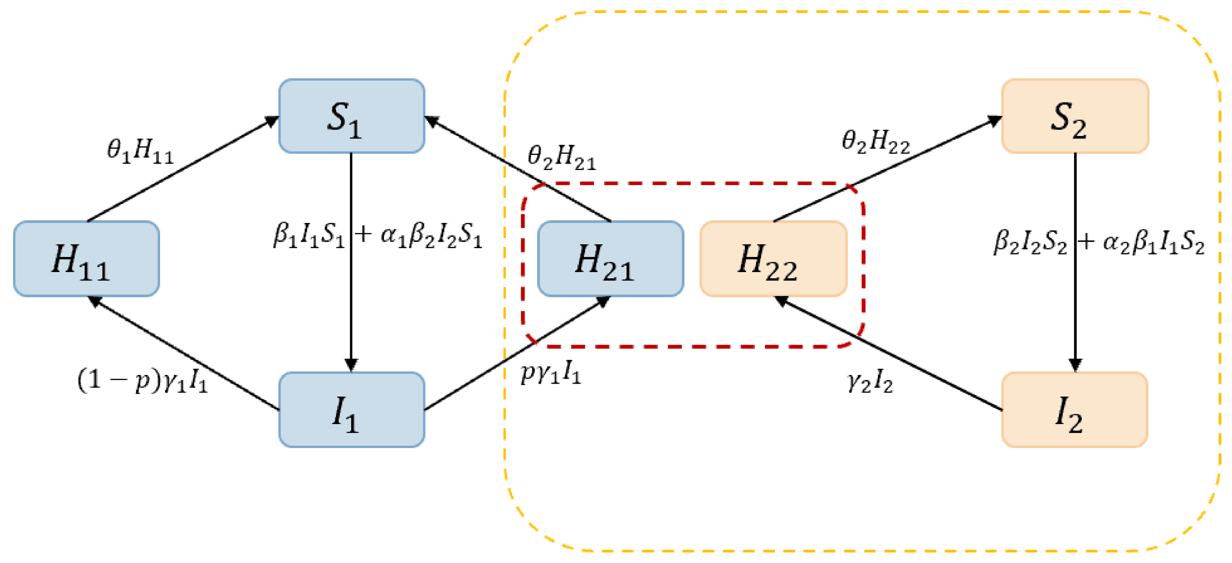

Figure 1 : Structure on the epidemic dynamics

	α_1	α_2	p
Weak lockdown type 1	0	+	+
Weak lockdown type 2	+	0	+
Strong lockdown	0	0	+
Complete lockdown	0	0	0

Table 1: Different types of lockdown

- The disease is not fatal;
- The community is composed of the peripheral area (area 1) and central area (area 2) which respectively have different qualities about the medical treatment for the disease;
- Susceptible individuals of one area can temporarily visit to the other area;
- Some infective individuals of the peripheral area (area 1) can get the medical treatment at the central area (area 2), for example, transported by ambulance;
- Recovered individual becomes susceptible again;
- The population size is constant in each area according to the epidemic dynamics.

Mathematical model

$$\begin{aligned}
\frac{dS_1}{dt} &= -\beta_1 I_1 S_1 - \alpha_1 \beta_2 I_2 S_1 + \theta_1 H_{11} + \theta_2 H_{21};\\
\frac{dI_1}{dt} &= \beta_1 I_1 S_1 + \alpha_1 \beta_2 I_2 S_1 - \gamma_1 I_1;\\
\frac{dH_{11}}{dt} &= (1-p)\gamma_1 I_1 - \theta_1 H_{11};\\
\frac{dH_{21}}{dt} &= p\gamma_1 I_1 - \theta_2 H_{21};\\
\frac{dS_2}{dt} &= -\beta_2 I_2 S_2 - \alpha_2 \beta_1 I_1 S_2 + \theta_2 H_{22};\\
\frac{dI_2}{dt} &= \beta_2 I_2 S_2 + \alpha_2 \beta_1 I_1 S_2 - \gamma_2 I_2;\\
\frac{dH_{22}}{dt} &= \gamma_2 I_2 - \theta_2 H_{22},
\end{aligned}$$

with $S_1 + I_1 + H_{11} + H_{21} = N_1$, $S_2 + I_2 + H_{22} = N_2$.

- S_i: Susceptible population density in area i.
- I_1: Infective population density in area i.
- H_{ij}: Population density of infected individuals who belong to area j and isolated under the medical treatment in area i.
- N_i: Total population size in area i.
- β_i: The infection coefficient at area i for the individual belonging to area i.
- $\alpha_i\beta_j$: The infection coefficient at area j for the individual belonging to area i during the temporary visit to area j, which is smaller than β_j $(0 < \alpha_i < 1)$.
- γ_i: The detection/isolation rate of the infective individual belonging to area i.
- θ_i: The discharge rate from the isolation under the medical treatment at area i.
- p: The proportion of infectives belonging to area 1 get the medical treatment in the central area $(0 \le p \le 1)$.

Non-dimensionalized system

$$\begin{aligned}
\frac{d\phi_1}{dt} &= -\mathscr{R}_0^r \gamma_1 \psi_1 \phi_1 - \mathscr{R}_0^c \gamma_2 \alpha_1 \psi_2 \phi_1 + \theta_1 \zeta_{11} + \theta_2 \zeta_{21};\\
\frac{d\psi_1}{dt} &= \mathscr{R}_0^r \gamma_1 \psi_1 \phi_1 + \mathscr{R}_0^c \gamma_2 \alpha_1 \psi_2 \phi_1 - \gamma_1 \psi_1;\\
\frac{d\zeta_{11}}{dt} &= (1-p)\gamma_1 \psi_1 - \theta_1 \zeta_{11};\\
\frac{d\zeta_{21}}{dt} &= p\gamma_1 \psi_1 - \theta_2 \zeta_{21};\\
\frac{d\phi_2}{dt} &= -\mathscr{R}_0^c \gamma_2 \psi_2 \phi_2 - \mathscr{R}_0^r \gamma_1 \alpha_2 \psi_1 \phi_2 + \theta_2 \zeta_{22};\\
\frac{d\psi_2}{dt} &= \mathscr{R}_0^c \gamma_2 \psi_2 \phi_2 + \mathscr{R}_0^r \gamma_1 \alpha_2 \psi_1 \phi_2 - \gamma_2 \psi_2;\\
\frac{d\zeta_{22}}{dt} &= \gamma_2 \psi_2 - \theta_2 \zeta_{22},
\end{aligned}$$

where $\phi_1 + \psi_1 + \zeta_{11} + \zeta_{21} = 1$, $\phi_2 + \psi_2 + \zeta_{22} = 1$, $\mathscr{R}_0^r = \beta_1 N_1/\gamma_1$, and $\mathscr{R}_0^c = \beta_2 N_2/\gamma_2$.

Epidemic scenarios

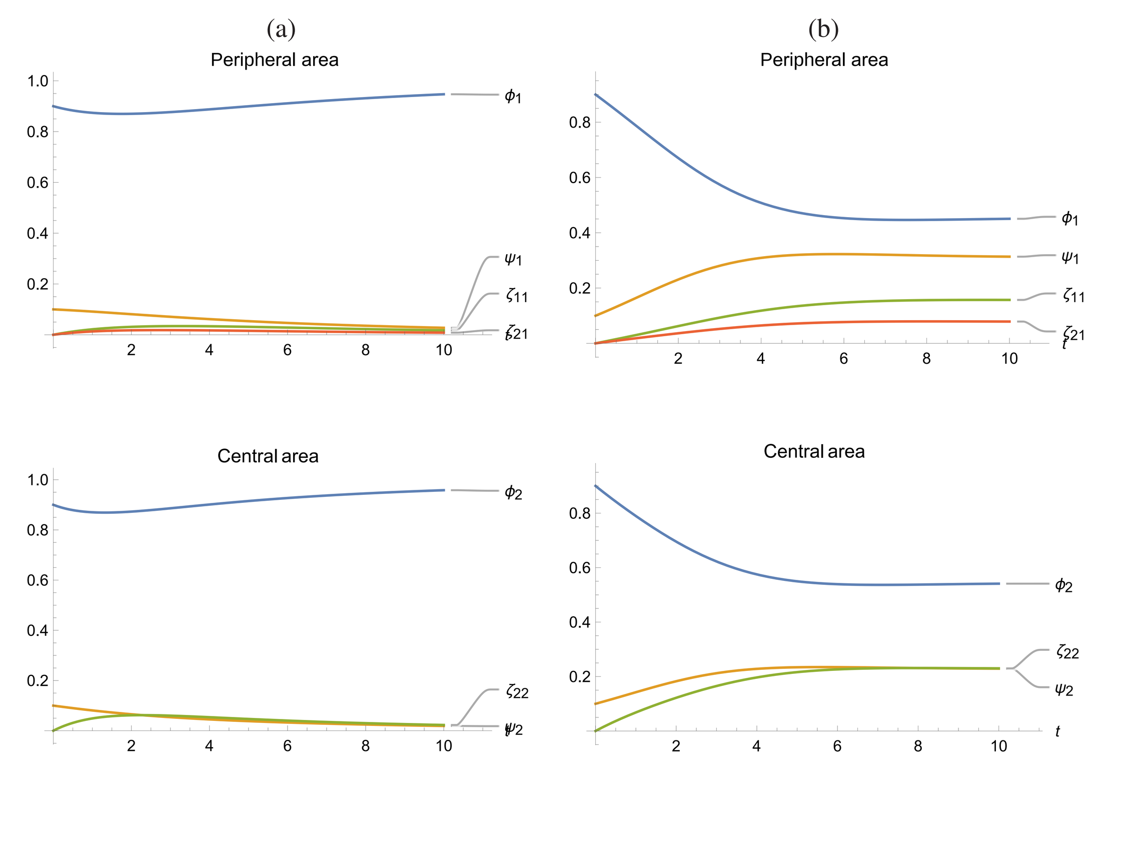

Figure 2 : $(\mathscr{R}_0^r, \mathscr{R}_0^c) =$ (a) $(0.4, 0.7)$; (b) $(1.5, 1.2)$. Commonly, $p = 0.4$; $\alpha_1 = \alpha_2 = 0.5$; $\theta_1 = 0.6$; $\theta_2 = 0.8$; $\gamma_1 = 0.5$; $\gamma_2 = 0.8$.

Endemic equilibrium

Theorem 1. *Disease-free equilibrium $E_0(1, 0, 0, 0, 1, 0, 0)$ is unstable if one of the following conditions is satisfied:*

(i) $\mathscr{R}_0^r \ge 1$;

(ii) $\mathscr{R}_0^c \ge 1$;

(iii) $\left(\frac{1}{\mathscr{R}_0^r} - 1\right)\left(\frac{1}{\mathscr{R}_0^c} - 1\right) < \alpha_1 \alpha_2$.

Lemma 1. *Endemic equilibrium $E^*(\phi_1^*, \psi_1^*, \zeta_{11}^*, \zeta_{21}^*, \phi_2^*, \psi_2^*, \zeta_{22}^*)$ uniquely exists if and only if one of the conditions $(i), (ii)$ and (iii) in Theorem 1 is satisfied, independently of which type of lockdown is adopted to the community.*

Theorem 2. *Under the strong lockdown with $\alpha_1 = \alpha_2 = 0$ or the complete lockdown with $\alpha_1 = \alpha_2 = p = 0$, the endemic equilibrium E^* is globally asymptotically stable when it exists.*

Endemic size

We define $\rho := N_1/N_2$. The endemic size is defined by $\Psi^* := (N_1 + N_2 - S_1^* - S_2^*)/(N_1 + N_2) = 1 - (\rho\phi_1^* + \phi_2^*)/(1+\rho)$. We can get the following formulas for the complete, strong, and weak (type 1 and 2) lockdowns by Ψ_c^*, Ψ_s^*, Ψ_{w1}^* and Ψ_{w2}^*:

$$\begin{aligned}
\Psi_c^* &= \frac{\rho}{(1+\rho)}\left(1 - \frac{1}{\mathscr{R}_0^r}\right) + \frac{1}{(1+\rho)}\left(1 - \frac{1}{\mathscr{R}_0^c}\right);\\
\Psi_s^* &= \frac{\rho}{(1+\rho)}\left(1 - \frac{1}{\mathscr{R}_0^r}\right) + \frac{1}{(1+\rho)}\left(1 - \frac{1}{\mathscr{R}_0^c}\right);\\
\Psi_{w1}^* &= \frac{\rho}{(1+\rho)}\left(1 - \frac{1}{\mathscr{R}_0^r}\right) + \frac{1}{(1+\rho)}(1 - \phi_2^*);\\
\Psi_{w2}^* &= \frac{\rho}{(1+\rho)}(1 - \phi_1^*) + \frac{1}{(1+\rho)}\left(1 - \frac{1}{\mathscr{R}_0^c}\right),
\end{aligned}$$

where ϕ_1^* is the smaller root of the following quadratic equation of x, which is less than $1/\mathscr{R}_0^r$:

$$\begin{aligned}
\mathscr{R}_0^r \gamma_1 \theta_1 \theta_2 x^2 - \Big\{ &(\mathscr{R}_0^r + 1)\gamma_1 \theta_1 \theta_2 + \mathscr{R}_0^c \alpha_1 \psi_2^* \gamma_2 \\
&\big[\theta_1 \theta_2 + (1-p)\gamma_1 \theta_2 + p\gamma_1 \theta_1\big] \Big\} x + \gamma_1 \theta_1 \theta_2 = 0
\end{aligned}$$

with $\psi_2^* = \theta_2[1 - (1/\mathscr{R}_0^c)]/(\theta_2 + \gamma_2)$. ϕ_2^* is the smaller root of the following quadratic equation of x, which is less than $1/\mathscr{R}_0^c$:

$$\begin{aligned}
\mathscr{R}_0^c \gamma_2 \theta_2 x^2 - \big[(\mathscr{R}_0^c + 1)\gamma_2 \theta_2 + \mathscr{R}_0^r \alpha_2 \psi_1^* \gamma_1 (\gamma_2 + \theta_2)\big] x \\
+ \gamma_2 \theta_2 = 0
\end{aligned}$$

with $\psi_1^* = \theta_1 \theta_2 [1 - (1/\mathscr{R}_0^r)]/[\theta_1 \theta_2 + (1-p)\gamma_1 \theta_2 + p\gamma_1 \theta_1]$.

The order of endemic size is $\Psi_c^* = \Psi_s^* < \Psi_{w\bullet}^*$.

Conclusion

- The complete and strong lockdown has the same endemic size, smaller than the weak lockdown.
- The weak lockdown with minimal restriction on mobility has the lowest efficiency in suppressing the spread of an epidemic.
- For the weak lockdown, more efficient is the prohibition of mobility for susceptible individuals from an area of high population density to that of low population density.
- When the hospital in the central area has a sufficiently longer isolation period than the peripheral area, free infectives under the strong lockdown are less than those under the complete lockdown.