# An Epidemic Dynamics Model with a Limited Capacity of Isolation for a Reinfectious Disease

Zhiqiong FU[1,*], Hiromi SENO[1]

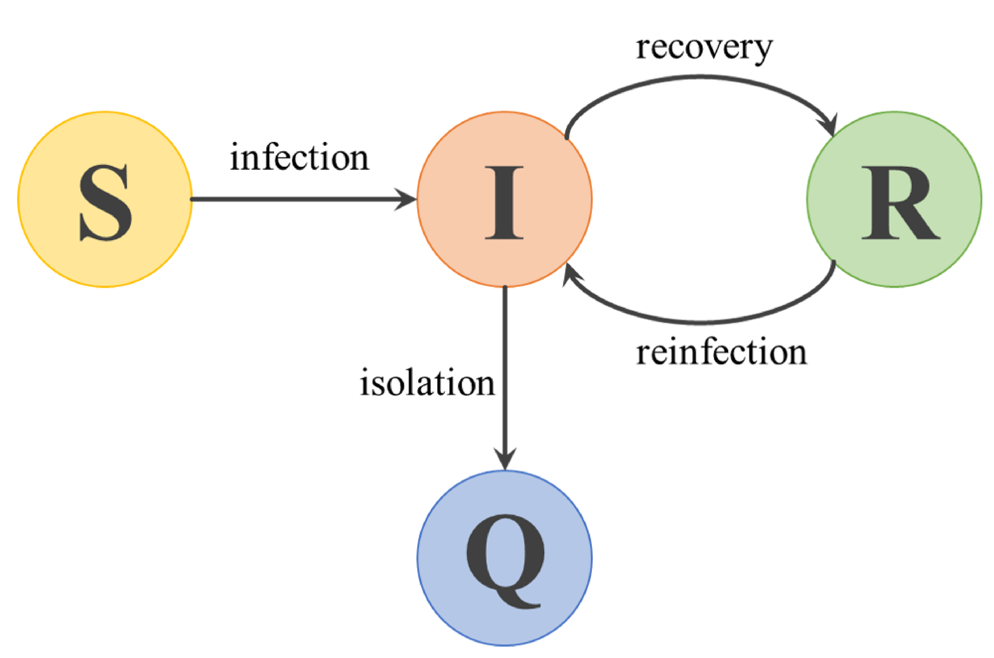
[1]Graduate School of Information Sciences, Tohoku University, Sendai, Japan

*fu.zhiqiong.t6@dc.tohoku.ac.jp

## What is considered

We consider an epidemic dynamics model with a system of ordinary differential equations for a reinfectious disease, in which a limited capacity of isolation is incorporated. By mathematical analyses on the model, we consider the relation of the limited isolation capacity to the epidemic consequence, and try to give some theoretical implications on the relevance of the isolation capacity with respect to the control of a disease spread in a community.

## Assumptions




- The total population size is constant, when any demographic change due to birth, death or migration is assumed to be negligible in the epidemic season;
- The recovered individual could be reinfected since a number of pathogenic variants could cause such a recurred infection;
- The isolated individual cannot contact others or be discharged in the epidemic season;
- The isolation will break down once it reaches the capacity.

## Isolation effective phase

This is the epidemic phase when the isolated subpopulation size $Q$ is less than the capacity $Q_{\max}$, and the isolation works with isolation rate $\sigma$.

$$\begin{aligned}
\frac{dS}{dt} &= -\beta\frac{I}{N-Q}S;\\
\frac{dI}{dt} &= \beta\frac{I}{N-Q}S + \varepsilon\beta\frac{I}{N-Q}R - \rho I - \sigma I;\\
\frac{dQ}{dt} &= \sigma I;\\
\frac{dR}{dt} &= \rho I - \varepsilon\beta\frac{I}{N-Q}R
\end{aligned}$$

## Isolation incapable phase

This is the epidemic phase when the isolated subpopulation size $Q$ has reached the capacity $Q_{\max}$, and the isolation breaks down to become incapable.

$$\begin{aligned}
\frac{dS}{dt} &= -\beta\frac{I}{N-Q_{\max}}S;\\
\frac{dI}{dt} &= \beta\frac{I}{N-Q_{\max}}S + \varepsilon\beta\frac{I}{N-Q_{\max}}R - \rho I;\\
\frac{dQ}{dt} &= 0;\\
\frac{dR}{dt} &= \rho I - \varepsilon\beta\frac{I}{N-Q_{\max}}R
\end{aligned}$$

## Epidemic dynamics model

$$\begin{aligned}
\frac{dS}{dt} &= -\beta\frac{I}{N-Q}S;\\
\frac{dI}{dt} &= \beta\frac{I}{N-Q}S + \varepsilon\beta\frac{I}{N-Q}R - \rho I - \Phi(Q,I);\\
\frac{dQ}{dt} &= \Phi(Q,I);\\
\frac{dR}{dt} &= \rho I - \varepsilon\beta\frac{I}{N-Q}R
\end{aligned}$$

with

$$\Phi(Q,I) = \begin{cases} \sigma I & \text{for } Q < Q_{\max};\\ 0 & \text{for } Q = Q_{\max}, \end{cases}$$

and the initial condition $(S(0), I(0), Q(0), R(0)) = (S_0, I_0, 0, 0)$ where $I_0 > 0$ and $S_0 = N - I_0 > 0$.

- $S$, $I$, $Q$ and $R$ are respectively the susceptible, infective, isolated, and recovered subpopulation sizes.
- $N$: Total population size in the community.
- $\beta$: The infection coefficient.
- $\varepsilon\beta$: The reinfection coefficient ($0 < \varepsilon < 1$).
- $\rho$: The natural recovery rate.
- $\Phi(Q, I)$: The isolation rate, where $Q_{\max}$ is the capacity of isolation, and $\sigma$ is the isolation rate at the isolation effective phase.

## Non-dimensionalized system

$$u := \frac{S}{N};\quad v := \frac{I}{N};\quad q := \frac{Q}{N};\quad w := \frac{R}{N};$$
$$\tau := (\rho+\sigma)t;\quad \gamma := \frac{\sigma}{\rho+\sigma};\quad q_{\max} := \frac{Q_{\max}}{N};$$
$$\mathscr{R}_0 := \frac{\beta}{\rho+\sigma};$$

$$\begin{aligned}
\frac{du}{d\tau} &= -\mathscr{R}_0\frac{v}{1-q}u;\\
\frac{dv}{d\tau} &= \mathscr{R}_0\frac{v}{1-q}u + \varepsilon\mathscr{R}_0\frac{v}{1-q}w - (1-\gamma)v - \phi(q,v);\\
\frac{dq}{d\tau} &= \phi(q,v);\\
\frac{dw}{d\tau} &= (1-\gamma)v - \varepsilon\mathscr{R}_0\frac{v}{1-q}w
\end{aligned}$$

with

$$\phi(q,v) = \begin{cases} \gamma v & \text{for } q < q_{\max};\\ 0 & \text{for } q = q_{\max}, \end{cases}$$

and the initial condition $(u(0), v(0), q(0), w(0)) = (u_0, v_0, 0, 0)$ where $v_0 > 0$ and $u_0 = 1 - v_0 > 0$.

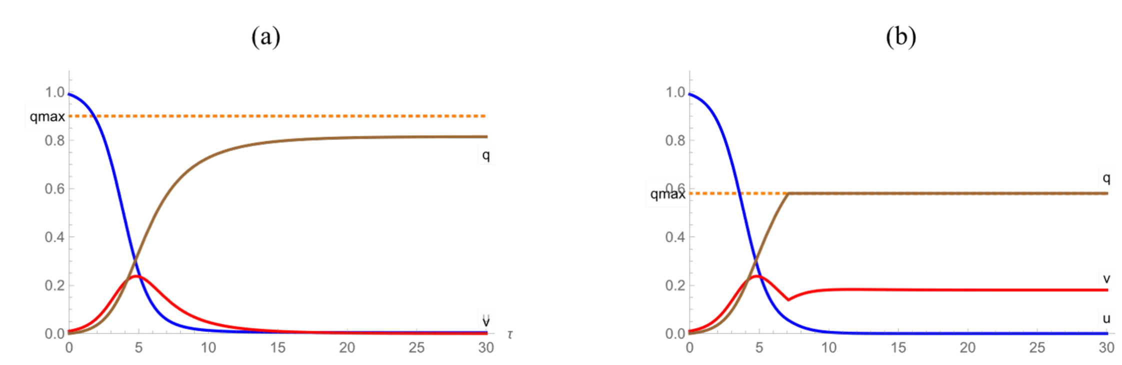



**Figure:** Temporal variations for (a) $q_{\max} = 0.9$; (b) $q_{\max} = 0.58$. Commonly, $u_0 = 0.99$; $\mathscr{R}_0 = 2.0$; $\varepsilon = 0.35$; $\gamma = 0.6$.

## Critical value of isolation capacity

The isolation reaches the capacity in a finite time and becomes incapable if and only if $q_{\max} < q_c$, where $q_c$ is the smallest positive root of the following equation:

$$u_0(1-q_c)^{\mathscr{R}_0/\gamma} = \frac{1-\varepsilon\mathscr{R}_0}{\gamma-\varepsilon\mathscr{R}_0}(1-q_c)\left\{1 - \frac{1-\gamma}{1-\varepsilon\mathscr{R}_0}(1-q_c)^{\varepsilon\mathscr{R}_0/\gamma-1}\right\},$$

where $q_c = 1$ for $\varepsilon\mathscr{R}_0 \geq 1$, and $q_c \in (0,1)$ for $\varepsilon\mathscr{R}_0 < 1$.

If $q_{\max} \geq q_c$, the isolation does not break down.

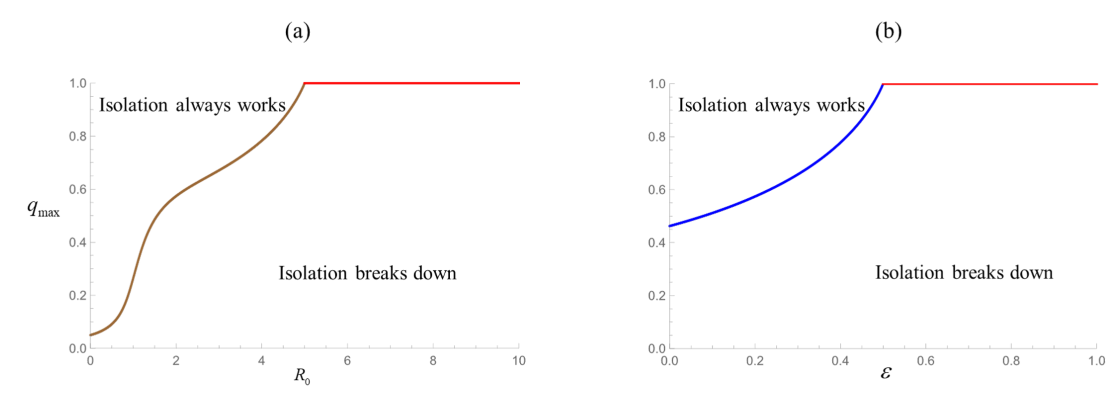



**Figure:** (a) $\mathscr{R}_0$-dependence and (b) $\varepsilon$-dependence of the critical value of the isolation capacity $q_c$. Numerically drawn for (a) $\varepsilon = 0.2$; (b) $\mathscr{R}_0 = 2$, and commonly with $\gamma = 0.5$; $u_0 = 0.9$.

## Final epidemic size $z_\infty := 1 - u_\infty$

For $q_{\max} \geq q_c$, $z_\infty = z_\infty^-$ such that

$$1 - z_\infty^- = \frac{1-\varepsilon\mathscr{R}_0}{\gamma-\varepsilon\mathscr{R}_0}\left(\frac{1-z_\infty^-}{u_0}\right)^{\gamma/\mathscr{R}_0} - \frac{1-\gamma}{\gamma-\varepsilon\mathscr{R}_0}\left(\frac{1-z_\infty^-}{u_0}\right)^{\varepsilon},$$

when $\varepsilon\mathscr{R}_0 < 1$.

For $q_{\max} < q_c$, $z_\infty = z_\infty^+$ such that

$$1 - z_\infty^+ = \frac{\varepsilon\mathscr{R}_0 - (1-\gamma)}{\varepsilon\mathscr{R}_0}(1-q_{\max}) + \left[\frac{\gamma}{\varepsilon\mathscr{R}_0}(1-q_{\max})^{(\gamma-\varepsilon\mathscr{R}_0)/\gamma} - 1\right]\frac{u_0^{-\varepsilon}(1-\gamma)(1-z_\infty^+)^{\varepsilon}}{\gamma-\varepsilon\mathscr{R}_0},$$

where $z_\infty^+ \in (1-u_0, 1)$ if $\varepsilon\mathscr{R}_0 < 1-\gamma$, and $z_\infty^+ = 1$ if $\varepsilon\mathscr{R}_0 \geq 1-\gamma$.

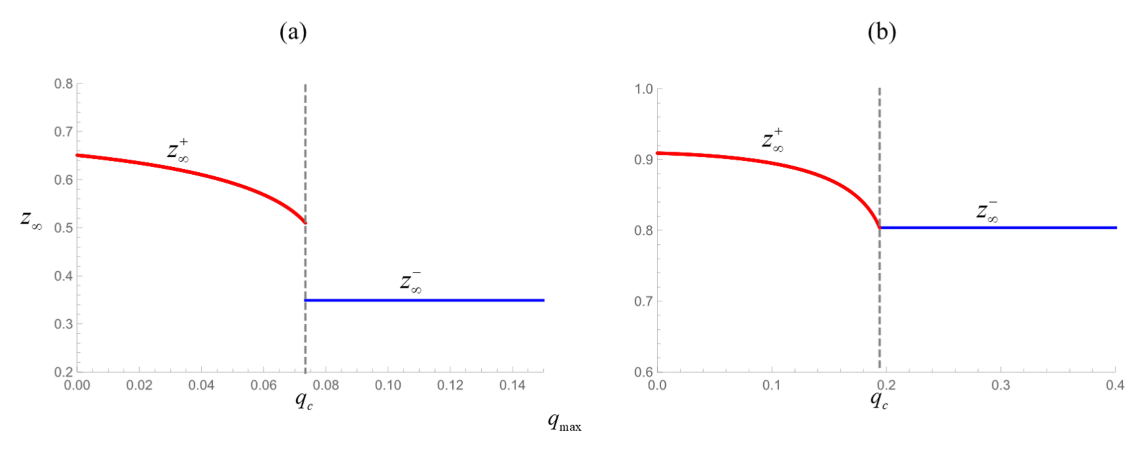



**Figure:** $q_{\max}$-dependence of the final epidemic size $z_\infty$. Numerically drawn for (a) $\mathscr{R}_0 = 1.1$; (b) $\mathscr{R}_0 = 1.5$, and commonly with $u_0 = 0.99$; $\varepsilon = 0.3$; $\gamma = 0.2$, where $q_c = 0.073, 0.194$ respectively.

## Endemic size

The endemic size $E_\infty := v^*$ is defined only with $\varepsilon\mathscr{R}_0 > 1-\gamma$ at the isolation incapable phase, where $v^*$ is the infective population at the endemic equilibrium:

$$E_\infty = \frac{\varepsilon\mathscr{R}_0 - (1-\gamma)}{\varepsilon\mathscr{R}_0}(1-q_{\max}).$$

## Conclusion

- The final epidemic size may be dramatically increased when the isolation reaches the capacity.
- The endemic size can be lessened by a larger capacity of isolation.
- A satisfactory isolation capacity to avoid the isolation breakdown depends on the nature of the disease spread.