An SIR+Q Model with Limited Capacity of Isolation

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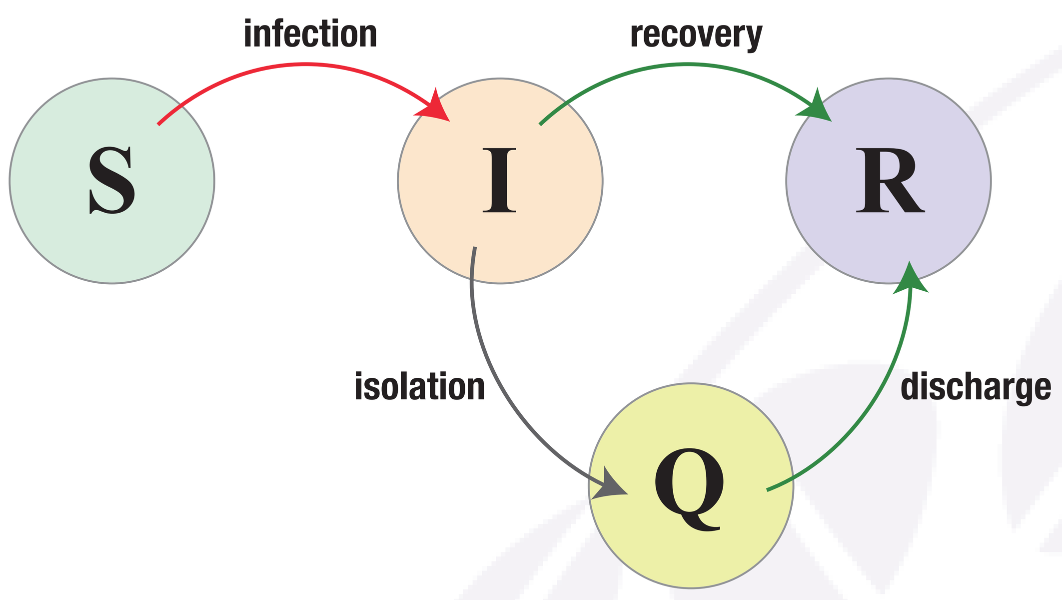
We analyze a mathematically reasonable and simplest epidemic dynamics model with a limited capacity of isolation, based on the SIR model, and obtain the mathematical results on how the final epidemic size depends on the capacity of isolation, which imply the existence of a critical capacity of isolation to suppress the epidemic size enough.

Assumptions



1. The considered epidemic dynamics is about a certain period (e.g., a season) during which any demographic change due to the birth, the death, and the migration is negligible with respect to the epidemic dynamics.
2. The infection occurs by the contact of susceptible individual to not only organic but also potentially inorganic subjects contaminated with the pathogen which causes the disease.
3. The isolation/hospitalization has a capacity beyond which the isolation is unavailable.
4. For mathematical simplicity, the availability of the isolation is constant independently of how many infectives are isolated as long as the isolation has not reached the capacity.

Isolation unsaturated phase

This phase indicates the situation that the isolated population size Q is less than $Q_{\max}$, so that the isolation works with a net rate given by σI. The epidemic dynamics is governed by

$$\frac{dS}{dt} = -\beta\frac{I}{N-Q}S;$$
$$\frac{dI}{dt} = \beta\frac{I}{N-Q}S - \rho I - \sigma I;$$
$$\frac{dQ}{dt} = \sigma I - \alpha Q;$$
$$\frac{dR}{dt} = \rho I + \alpha Q.$$

Isolation saturated phase

This phase indicates the situation that the isolated population size Q has reached $Q_{\max}$. However, the isolation is kept working since there are always some individuals discharged from the isolation, and then the isolation becomes capable for the corresponding vacancy.

$$\frac{dS}{dt} = -\beta\frac{I}{N-Q_{\max}}S;$$
$$\frac{dI}{dt} = \beta\frac{I}{N-Q_{\max}}S - \rho I - \min\left[\sigma I,\, \alpha Q_{\max}\right];$$
$$\frac{dQ}{dt} = \min\left[\sigma I,\, \alpha Q_{\max}\right] - \alpha Q_{\max};$$
$$\frac{dR}{dt} = \rho I + \alpha Q_{\max}.$$

Epidemic dynamics model

$$\frac{dS}{dt} = -\beta\frac{I}{N-Q}S;$$
$$\frac{dI}{dt} = \beta\frac{I}{N-Q}S - \rho I - \Phi(Q,I);$$
$$\frac{dQ}{dt} = \Phi(Q,I) - \alpha Q;$$
$$\frac{dR}{dt} = \rho I + \alpha Q$$

with

$$\Phi(Q,I) = \begin{cases} \sigma I & \text{for } Q < Q_{\max}; \\ \min\left[\sigma I,\, \alpha Q_{\max}\right] & \text{for } Q = Q_{\max} \end{cases}$$

and the initial condition $(S(0), I(0), Q(0), R(0)) = (S_0, I_0, 0, 0)$ where $I_0 > 0$ and $S_0 = N - I_0 > 0$.

Non-dimensionalized system

$$\tau := (\rho+\sigma)t; \quad u := \frac{S}{N}; \quad v := \frac{I}{N}; \quad q := \frac{Q}{N}; \quad w := \frac{R}{N};$$
$$\mathscr{R}_0 := \frac{\beta}{\rho+\sigma}; \quad a := \frac{\alpha}{\rho+\sigma}; \quad \gamma := \frac{\sigma}{\rho+\sigma}; \quad q_{\max} := \frac{Q_{\max}}{N},$$

where $\mathscr{R}_0 := \beta/(\rho+\sigma)$ is the basic reproduction number for our model.

$$\frac{du}{d\tau} = -\mathscr{R}_0\frac{v}{1-q}u;$$
$$\frac{dv}{d\tau} = \mathscr{R}_0\frac{v}{1-q}u - (1-\gamma)v - \phi(q,v);$$
$$\frac{dq}{d\tau} = \phi(q,v) - aq;$$
$$\frac{dw}{d\tau} = (1-\gamma)v + aq$$

with

$$\phi(q,v) = \begin{cases} \gamma v & \text{for } q < q_{\max}; \\ \min\left[\gamma v,\, a q_{\max}\right] & \text{for } q = q_{\max}, \end{cases}$$

and the initial condition $(u(0), v(0), q(0), w(0)) = (u_0, v_0, 0, 0)$ where $v_0 > 0$ and $u_0 = 1 - v_0 > 0$.

Model with no discharge ($a = 0$)

$$\frac{du}{d\tau} = -\mathscr{R}_0\frac{v}{1-q}u;$$
$$\frac{dv}{d\tau} = \mathscr{R}_0\frac{v}{1-q}u - (1-\gamma)v - \phi(q,v);$$
$$\frac{dq}{d\tau} = \phi(q,v);$$
$$\frac{dw}{d\tau} = (1-\gamma)v$$

with

$$\phi(q,v) = \begin{cases} \gamma v & \text{for } q < q_{\max}; \\ 0 & \text{for } q = q_{\max}. \end{cases}$$

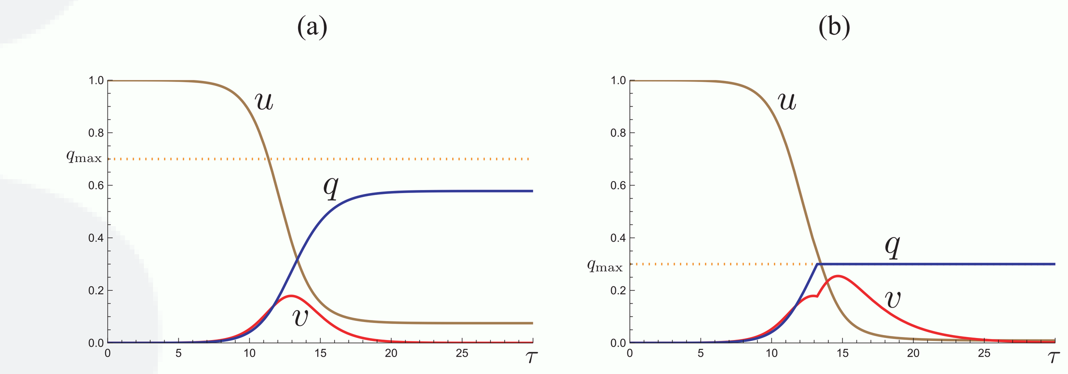


Figure: Temporal variations for (a) $q_{\max} = 0.7$; (b) $q_{\max} = 0.3$, commonly with $\mathscr{R}_0 = 1.875$; $\gamma = 0.625$; $a = 0$; $u_0 = 0.99999$.

Critical isolation capacity ($a = 0$)

The isolation reaches the capacity in a finite time on the way of epidemic process if and only if $q_{\max} < q_c$, where q_c is uniquely determined by the positive root of the following equation:

$$1 - \frac{q_c}{\gamma} = u_0(1-q_c)^{\mathscr{R}_0/\gamma}.$$

If $q_{\max} \geq q_c$, the isolation never reaches the capacity, and is always available.

The critical value for the isolation capacity q_c is monotonically increasing in terms of $\mathscr{R}_0$ and γ, necessarily satisfying that $q_c < \gamma$.

Final epidemic size ($a = 0$)

The final epidemic size $z_\infty := q_\infty + w_\infty$ is uniquely determined by the following equations:

$$u_0(1-\gamma z_\infty^-)^{\mathscr{R}_0/\gamma} = 1 - z_\infty^-;$$
$$\frac{\mathscr{R}_0}{\gamma}\left\{\frac{1}{1-\gamma}\cdot\frac{q_{\max}}{1-q_{\max}} + \ln(1-q_{\max})\right\} = \ln(1-z_\infty^+) - \ln u_0 + \frac{\mathscr{R}_0}{1-\gamma}\cdot\frac{z_\infty^+}{1-q_{\max}},$$

where $z_\infty = z_\infty^- \in (1-u_0, 1)$ for $q_{\max} \geq q_c$ and $z_\infty = z_\infty^+ \in (1-u(t^*), 1)$ for $q_{\max} < q_c$ with $u(t^*) = (1-q_{\max})^{\mathscr{R}_0/\gamma}u_0$.

Discontinuity of the final epidemic size ($a = 0$)

The final epidemic size has a discontinuous change at $q = q_c$ such that

$$z_\infty^\dagger := \lim_{q_{\max}\to q_c-0} z_\infty^+ > z_\infty^-$$

if and only if

$$\mathscr{R}_0 > 1-\gamma \quad \text{and} \quad u_0 > \frac{1-\gamma}{\mathscr{R}_0}\left(\frac{1}{1-\gamma} - \frac{\gamma}{\mathscr{R}_0}\right)^{\mathscr{R}_0/\gamma - 1}.$$

Otherwise, it holds that $z_\infty^\dagger = z_\infty^-$.

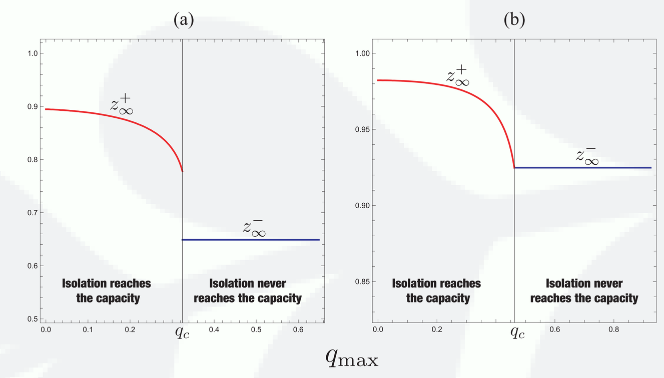


Figure: $q_{\max}$-dependence of the final epidemic size z_∞. (a) $\mathscr{R}_0 = 1.2$; (b) $\mathscr{R}_0 = 2.0$, commonly with $\gamma = 0.5$; $u_0 = 0.9$.

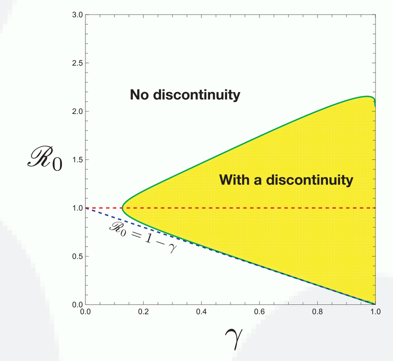


Figure: $(\gamma, \mathscr{R}_0)$-region for the discontinuity of the final epidemic size z_∞ at $q = q_c$. Numerically drawn with $u_0 = 0.99$.

Model with discharge ($a > 0$)

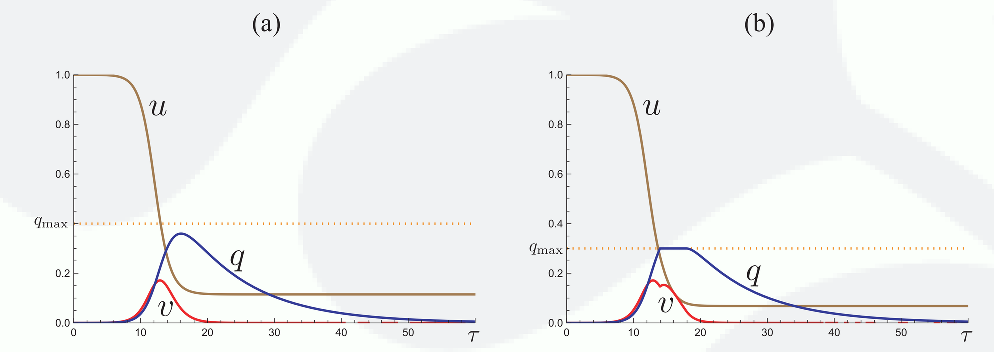


Figure: Temporal variations for (a) $q_{\max} = 0.4$; (b) $q_{\max} = 0.3$, commonly with $\mathscr{R}_0 = 1.875$; $\gamma = 0.625$; $a = 0.1$; $u_0 = 0.99999$.

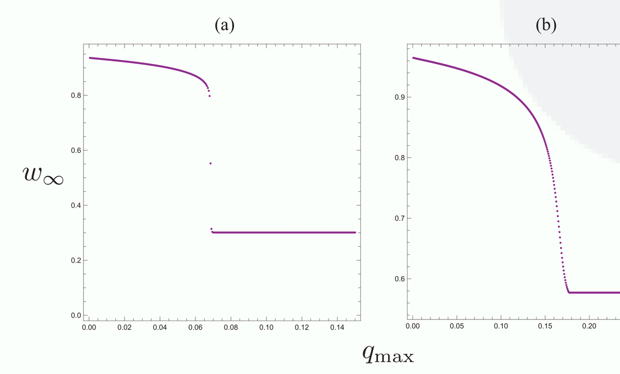


Figure: $q_{\max}$-dependence of the final epidemic size w_∞. (a) $\mathscr{R}_0 = 1.1$; (b) $\mathscr{R}_0 = 1.3$, commonly with $\gamma = 0.625$; $a = 0.1$; $u_0 = 0.99$, where $q_c = 0.266, 0.439$ respectively for the model with no discharge.

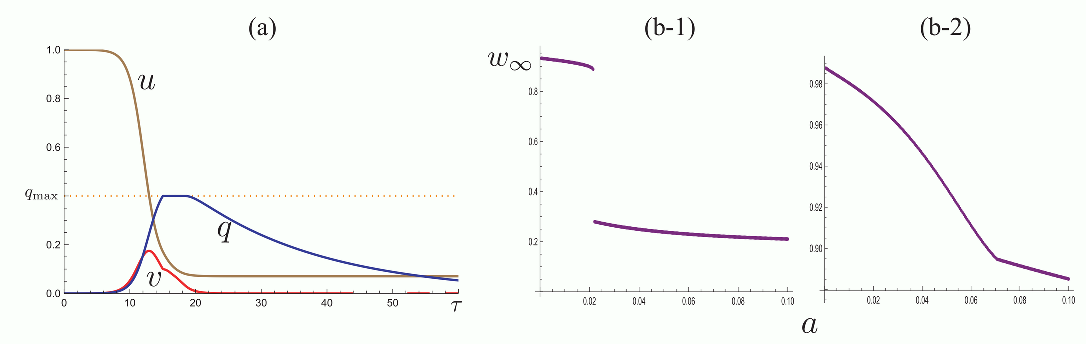


Figure: (a) a temporal variation with $a = 0.05$; (b) a-dependence of the final epidemic size w_∞. Numerically drawn with $(\mathscr{R}_0, q_{\max}) =$ (a, b-2) $(1.875, 0.4)$; (b-1) $(1.1, 0.1)$, and commonly $\gamma = 0.625$; $u_0 = 0.99999$, where $q_c =$ (a, b-2) 0.5780; (b-1) 0.2345 for the model with no discharge.

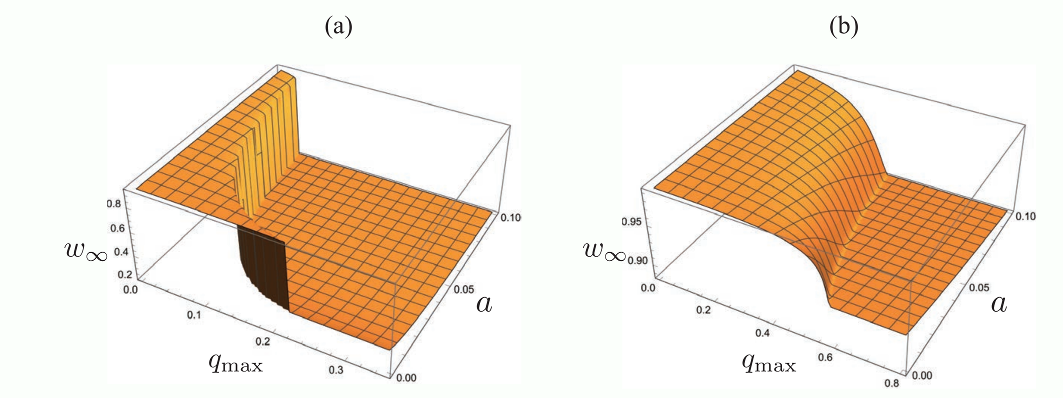


Figure: $(q_{\max}, a)$-dependence of the final epidemic size w_∞. Numerically drawn with $\mathscr{R}_0 =$ (a) 1.1; (b) 1.875, and commonly $\gamma = 0.625$; $u_0 = 0.99999$, where $q_c =$ (a) 0.2345; (b) 0.5780 for the model with no discharge.

Conclusion

- There exists a critical value for the isolation capacity such that the epidemic size becomes significantly large if the capacity is below it. The critical value is positively correlated to the basic reproduction number.
- There exists a finite supremum about the critical value for the isolation capacity such that the isolation never reaches the capacity if the capacity is not below it. The supremum is determined only by the recovery and isolation rates which could be improved by the advance in the medical treatment and service.
- Too large capacity of isolation could not be beneficial at all for the suppression of the epidemic size.
- The effective treatment to shorten the isolation period could significantly reduce the epidemic size.

Reference: Ahmad, I. and Seno, H., An epidemic dynamics model with limited isolation capacity, *(in preparation)*.

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