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ORIGINAL ARTICLE

An epidemic dynamics model with limited isolation capacity

Ishfaq Ahmad[1] · **Hiromi Seno**[1]

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Abstract
We consider a modified SIR model with a four-dimensional system of ordinary differential equations to consider the influence of a limited isolation capacity on the final epidemic size defined as the total number of individuals who experienced the disease at the end of an epidemic season. We derive the necessary and sufficient condition that the isolation reaches the capacity in a finite time on the way of the epidemic process, and show that the final epidemic size is monotonically decreasing in terms of the isolation capacity. We find further that the final epidemic size could have a discontinuous change at the critical value of isolation capacity below which the isolation reaches the capacity in a finite time. Our results imply that the breakdown of isolation with a limited capacity would cause a drastic increase of the epidemic size. Insufficient capacity of the isolation could lead to an unexpectedly severe epidemic situation, while such a severity would be avoidable with the sufficient isolation capacity.

Keywords Epidemic dynamics · Mathematical model · Ordinary differential equations · Isolation · Final epidemic size

Mathematics Subject Classification 92B99 · 92D30 · 92D25 · 91D99 · 00A71

Introduction

Infectious diseases have been an enemy of the human population that has caused the death of numerous humans. Even in recent history, infectious diseases have affected population growth. In general, such infectious diseases can be vanished and reappear in the future. For example, we had Spanish flu (1918–1919) and Black Deaths (1346–1350) which started in Asia, entered Europe, and reappeared for three decades before finally being eliminated (Brauer 2017). Epidemiologists are always concerned about the outbreak of diseases, while human daily activities could increase their worries (Hara and Yamaguchi 2021; Nagata et al. 2021).

Mathematical modeling of epidemic dynamics could play an important role to discuss how an infectious disease could spread, the expected duration of the epidemic, the expected number of infected, and the epidemiological indices to characterize the epidemic severity, including the basic reproductive number. The early work by Kermack and McKendrick in 1927 is regarded as one of the important origins of mathematical modeling on epidemic dynamics and has been widely applied for a variety of epidemic problems (Kermack and McKendrick 1927).

To reduce the risk of the spread of an infectious disease in the community, the strategies of quarantine, isolation, vaccination, and treatment as the policy for the public health are important. To manage various kinds of infectious diseases like severe acute respiratory syndrome, plague, smallpox, cholera, yellow fever, influenza virus, and SARS-CoV-2, the quarantine, isolation, and vaccination are primary. Martcheva (2015) gives a summary of such policies used to manage the spread of infectious diseases. Actually in the pandemic of COVID-19, there have been different policies for the public health from place to place (for example, Pearce et al. (2020), Mendez-Brito et al. (2021), Unruh et al. (2022)).

Until now a lot of works have been done with mathematical models including the isolation process for the purpose to consider its contribution to the suppression of a disease spread (for example, Feng and Thieme (1995), Brauer and

✉ Ishfaq Ahmad
ishfaqmaths899@gmail.com

Hiromi Seno
seno@math.is.tohoku.ac.jp

[1] Department of Computer and Mathematical Sciences, Graduate School of Information Sciences, Tohoku University, Aramaki-Aza-Aoba 6-3-09, Aoba-ku, Sendai, Miyagi 980-8579, Japan

Castillo-Chavez (2012), Chowell et al. (2016) and references therein). Hethcote et al. (2002) proposed SIR+Q and SIQS mathematical models introduced an isolated state (Q) with three forms of incidence. In their SIR+Q model with a quarantine-adjusted incidence, the endemic equilibrium is an unstable spiral for a set of parameter values, and a periodic solution arises with Hopf bifurcation. Castillo-Chavez et al. (2003) considered the mathematical model for the purpose to discuss whether the quarantine/isolation could manage the SARS for a limited time frame within a single outbreak. Their model implied that the quarantine/isolation could significantly reduce the size of SARS outbreak. Vivas-Barber et al. (2014) considered an SIR+Q model with the perfect isolation and an asymptomatic state and got a damped oscillation of the infective population size.

In many countries, there has been a shortage of medical resources under the outbreak of COVID-19 (Unruh et al. 2022). In recent times, some works using mathematical models consider how the limited medical resources could affect the transmission and management of an infectious disease (Abdelrazec et al. 2016; Wang et al. 2018; Saha and Samanta 2019; Sepulaveda-Salcedo et al. 2020; Zhao et al. 2020). Hu et al. (2022) considered an SAIQR mathematical model to consider the transmission dynamics of SARS-CoV-2 with a limited medical resource under the human migration between two regions, taking account of the asymptomatic state (A). Their results imply that making the basic reproduction number below 1 is not sufficient in order to manage the outbreak of COVID-19, and it should be significantly below 1. A local outbreak may occur when the medical resources are limited, even when the disease is indexed by a reproduction number below 1.

Even the quarantine/isolation may be perfect or imperfect depending upon the epidemic nature and policies implemented by the community. Erdem et al. (2017) considered the case of imperfect quarantine/isolation and found a periodic or damped oscillation that indicates recurring outbreaks, depending on the quarantine effectiveness. It is obvious that the isolation requires a specific space with highly organized conditions to keep the infected individuals away from the other community members, so that there must be a certain capacity for it. With its insufficient capacity, the isolation strategy may break down at a finite time on the way of epidemic process. Amador and Gomez-Corral (2020) considered a stochastic SIQS model with susceptible, infected, and two quarantine states in which the quarantine has a limited capacity. Their numerical calculation showed a case where the quarantine compartment tends to become full before the outbreak ends, whereas they did not clarify the exact condition for such a case since their numerics were aimed not to discuss the biological meanings of the results but to investigate the mathematical nature of their stochastic model. Since the isolation must be one of factors to determine the epidemic consequence even if it breaks down at a certain moment under the disease spread in the community, we are interested in how the final epidemic size depends on the isolation capacity.

In this paper, we focus on the relation of a limited isolation capacity to the final epidemic size for a simplest SIR+Q model. We derive the equation that determines the final epidemic size, respectively, when the isolation never reaches the capacity at any time and when the isolation reaches the capacity in a finite time, and then discuss how the limited isolation capacity could influence the final epidemic size. We can find the condition that the isolation reaches the capacity in a finite time on the way of the epidemic process. Further we find that the final epidemic size could not be necessarily continuous in terms of the isolation capacity, and derive the condition of the continuity and discontinuity at the critical value of isolation capacity below which the isolation reaches the capacity in a finite time. Our theoretical results would highlight the importance of satisfactory infrastructure for the public health as indicated by Unruh et al. (2022) on the social response to the COVID-19 pandemic. Since the satisfactory infrastructure for the public health needs a sufficient social investment, those arguments on our model would imply a difficulty of the management of even isolation policy against an infectious disease spreading in a community too.

Assumptions and modeling

We consider an epidemic dynamics in a season, which consists of susceptible, infective, isolated, and recovered individuals. We assume the followings for our modeling:

- The total population size of the community is constant, ignoring any demographic change with birth, death, and migration in a given epidemic season.
- Isolated individuals cannot contact any other in the community.
- Any isolated individual is not discharged in the season.
- The isolation capacity is limited. When the isolation reaches the capacity, it breaks down and becomes incapable.

Following the last assumption, the epidemic dynamics may contain two phases: *isolation effective phase* and *isolation incapable phase*. The isolation is available at the isolation effective phase, while it is ceased at the isolation incapable phase since it has reached the capacity.

With the above assumptions, we consider the following SIR+Q model in this paper:

$$\begin{aligned}\frac{dS}{dt} &= -\beta\frac{I}{N-Q}S;\\ \frac{dI}{dt} &= \beta\frac{I}{N-Q}S-\gamma I-\sigma(Q)I;\\ \frac{dQ}{dt} &= \sigma(Q)I;\\ \frac{dR}{dt} &= \gamma I\end{aligned} \tag{1}$$

with

$$\sigma(Q)=\begin{cases}\sigma_0 & Q<Q_{\max};\\ 0 & Q=Q_{\max}\end{cases}$$

and the initial condition $(S(0), I(0), Q(0), R(0)) = (S_0, I_0, 0, 0)$. The variables S, I, Q, and R denote the sizes of susceptible, infective, isolated, and recovered subpopulations, respectively. The total population size of the community is denoted by a positive constant N, and it is satisfied that $S(t)+I(t)+Q(t)+R(t)=N$ for any $t \geq 0$. Hence it holds that $S_0+I_0=N$. The individual state transition according to the epidemic dynamics is schematically shown in Fig. 1.

Every parameter is positive. The parameter γ denotes the recovery rate of infective individual. The disease transmission follows the frequency-dependent infection force with the infection coefficient β. Since the subpopulation size of free individuals is given by $N-Q$, the net incidence rate is given by $\beta SI/(N-Q)$ in our modeling. The piece-wise function $\sigma(Q)$ denotes the isolation rate of infected individual. Parameter σ_0 is the isolation rate at the isolation effective phase, which represents the efficiency of quarantine operation to detect and isolate an infective.

The parameter $Q_{\max}$ denotes the capacity of isolation. As a reasonable setup, we assume that $Q_{\max} < N$. As long as the isolated subpopulation size Q is less than the capacity $Q_{\max}$, the isolation is available, and the epidemic dynamics is at the isolation effective phase with $\sigma(Q)=\sigma_0$. Once Q reaches $Q_{\max}$, the isolation becomes ceased after it. Then the epidemic dynamics enters in the isolation incapable phase with $\sigma(Q)=0$, as numerically exemplified in Fig. 2b. As the above assumption, the isolated subpopulation size Q remains $Q_{\max}$ at the isolation incapable phase since any isolated individual is not discharged from the isolation. In contrast, as numerically exemplified in Fig. 2a, for a sufficiently large capacity $Q_{\max}$, the epidemic dynamics always remains at the isolation effective phase with $\sigma(Q)=\sigma_0$, since the isolation never reaches the capacity.

We now derive the following non-dimensionalized system mathematically equivalent to model (1) with the transformation of variables to the proportions in the community such as $s(t)=S(t)/N$, $i(t)=I(t)/N$, $q(t)=Q(t)/N$, and $r(t)=R(t)/N$. Further we introduce the dimensionless time $\hat{t}=t/\tau$, where $\tau := 1/(\gamma+\sigma_0)$ is the expected duration of the infectivity for an infective at the isolation effective phase.

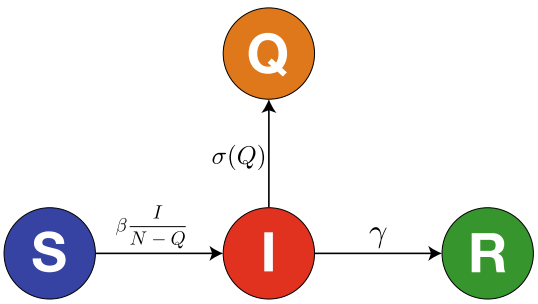

Fig. 1 Scheme for epidemic dynamics model (1)

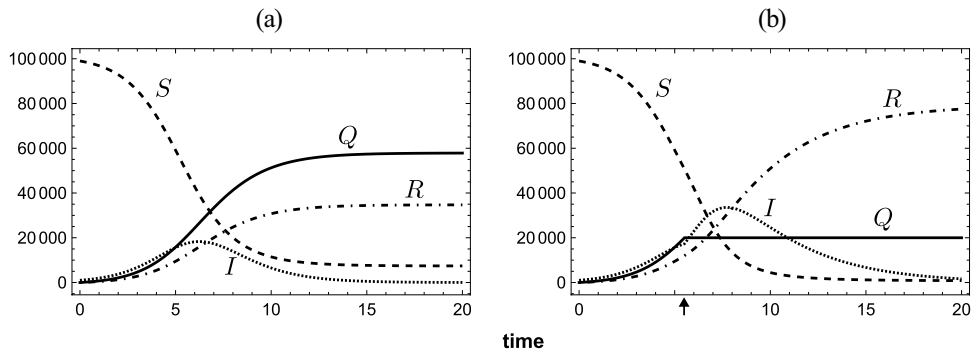

Fig. 2 Numerical examples for the temporal variation of SIR+Q model (1). **a** $Q_{\max}=70000$; **b** $Q_{\max}=20000$, $\beta=1.5$; $\gamma=0.3$; $\sigma_0=0.5$; $N=100000$; $(S_0, I_0, Q_0, R_0)=(99000, 1000, 0, 0)$. In **a**, the isolation never reaches the capacity, while it reaches the capacity at a moment indicated by an arrow in **b**. At the moment, the epidemic dynamics switches from the isolation effective phase to the isolation incapable phase

$$\begin{aligned}
\frac{ds}{d\hat{t}} &= -\mathscr{R}_0 \frac{is}{1-q};\\
\frac{di}{d\hat{t}} &= \mathscr{R}_0 \frac{is}{1-q} - \hat{\gamma} i - \hat{\sigma}(q) i;\\
\frac{dq}{d\hat{t}} &= \hat{\sigma}(q) i;\\
\frac{dr}{d\hat{t}} &= \hat{\gamma} i
\end{aligned} \tag{2}$$

with $\mathscr{R}_0 = \beta/(\gamma + \sigma_0)$ and

$$\hat{\sigma}(q) = \begin{cases} \hat{\sigma}_0 & q < q_{\max}; \\ 0 & q = q_{\max}, \end{cases}$$

where $\hat{\gamma} = \gamma/(\gamma + \sigma_0)$, $\hat{\sigma}_0 = \sigma_0/(\gamma + \sigma_0)$, and $q_{\max} = Q_{\max}/N$ $(0 < q_{\max} < 1)$. The initial condition is expressed as $(s(0), i(0), q(0), r(0)) = (s_0, i_0, 0, 0)$ with $s_0 = S_0/N$ and $i_0 = I_0/N$. It is satisfied that $s_0 + i_0 = 1$ and $s(\hat{t}) + i(\hat{t}) + q(\hat{t}) + r(\hat{t}) = 1$ for any $\hat{t} > 0$.

Remark that, aside from the isolated state with a limited capacity, the epidemic dynamics is fundamentally governed by the one-way state transition as an SIR model, so that necessarily $I(t) \to 0$ as $t \to \infty$, that is, $i(\hat{t}) \to 0$ as $\hat{t} \to \infty$ as well as the simple Kermack-McKendrick SIR model (Brauer et al. 2008; Brauer and Castillo-Chavez 2012; Martcheva 2015).

Basic reproduction number

The parameter $\mathscr{R}_0$ in (2) corresponds to the basic reproduction number for the epidemic dynamics governed by (1). The definition of basic reproduction number in biological context is the expected number of new cases produced by a single infective individual in a community where the infective individual contacts only susceptible individuals until its infectivity is lost (for the recent review about the definition, the translation, and the practical application, see Delamater et al. (2019)). For the mathematical derivation of the basic reproduction number for our model (1), we may use a fundamental way used in Brauer et al. (2008) and can get $\mathscr{R}_0 = \beta/(\gamma + \sigma_0)$, where $1/(\gamma + \sigma_0)$ is the expected duration of infectivity for an infective at the isolation effective phase, and then we find that β in our model corresponds to the supremum of the expected number of new cases produced by an infective per unit time (Appendix A). If $\mathscr{R}_0 < 1$, the disease dies out from the initial condition with sufficiently small number of infectives, while, if $\mathscr{R}_0 > 1$, the disease spreads at least at the initial stage of epidemic dynamics in the community.

Conserved quantity for each phase

We can find the conserved quantity for the epidemic dynamics at the isolation effective and incapable phases, respectively (Appendix B).

At the isolation effective phase

$$s(\hat{t}) + i(\hat{t}) = -\frac{\gamma}{\sigma_0} + \left\{ \frac{s(\hat{t})}{s_0} \right\}^{\sigma_0/\beta} \left(1 + \frac{\gamma}{\sigma_0} \right). \tag{3}$$

Equation (3) gives a relation satisfied by the solution of (2) with $\hat{\sigma}(q) = \hat{\sigma}_0$ for any $\hat{t} \geq 0$ at the isolation effective phase.

At the isolation incapable phase

$$s(\hat{t}) + i(\hat{t}) = s(t^\star) + i(t^\star) + \frac{\gamma}{\beta}(1 - q_{\max}) \ln \frac{s(\hat{t})}{s(t^\star)} \qquad (\hat{t} > t^\star), \tag{4}$$

where $\hat{t} = t^\star$ is supposed as the moment at which the isolation reaches the capacity, that is, when the isolation strategy breaks down due to an insufficient isolation capacity, and then the dynamics switches from the isolation effective phase to the isolation incapable phase. Equation (4) is satisfied by the solution of (2) with $\hat{\sigma}(q) = 0$ for any $\hat{t} > t^\star$ at the isolation incapable phase. Remark that, supposed that the isolation reaches the capacity at $\hat{t} = t^\star$, equation (3) holds for $\hat{t} \leq t^\star$ about system (2) with $\hat{\sigma}(q) = \hat{\sigma}_0$.

Critical value of the isolation capacity q_c

We obtain the following theorem and corollaries about the condition that the isolation reaches the capacity in a finite time on the way of epidemic process (Appendix C):

Theorem 5.1 *The isolation reaches the capacity in a finite time on the way of epidemic process if and only if*

$$1 - q_{\max}\left(1 + \frac{\gamma}{\sigma_0} \right) > s_0 (1 - q_{\max})^{\beta/\sigma_0}. \tag{5}$$

Corollary 5.1.1 *The isolation reaches the capacity in a finite time on the way of epidemic process if and only if $q_{\max} < q_c$, where q_c is the critical value of the isolation capacity and uniquely determined by the positive root of the following equation*:

$$1 - q_c\left(1 + \frac{\gamma}{\sigma_0} \right) = s_0 (1 - q_c)^{\beta/\sigma_0}. \tag{6}$$

If $q_{\max} \geq q_c$, the isolation never reaches the capacity and is always available.

Corollary 5.1.2 *The isolation reaches the capacity in a finite time on the way of epidemic process only if* $q_{\max} < 1/(1+\gamma/\sigma_0)$.

In other words, Corollary 5.1.2 indicates that the isolation never reaches the capacity for $q_{\max} \geq 1/(1+\gamma/\sigma_0)$. Hence, from Corollary 5.1.1, we find that necessarily $q_c < 1/(1+\gamma/\sigma_0)$.

From equation (6), we can easily find that the critical value of the isolation capacity q_c is monotonically increasing in terms of the infection coefficient β. The higher likelihood of infection leads to the demand of a larger capacity of isolation to avoid its breakdown in the epidemic dynamics. In contrast, we can find as well that q_c is monotonically decreasing in terms of the initial susceptible size s_0 and the recovery rate γ.

The smaller s_0 means the larger initial infective size i_0. Hence this result indicates that the larger isolation capacity is required for the larger initial infective size in order to avoid its saturation, that is, its breakdown. This is because the larger initial infective size must lead to a larger number of secondary cases which is more likely to cause the saturation of isolation.

As the patient can recover after a shorter expected duration of infectivity, defined by $1/\gamma$, the isolation capacity to avoid its saturation is smaller. Since the shorter duration of infectivity leads to the smaller infective subpopulation size, the increase of isolated subpopulation must be slower, so that the isolation capacity could be smaller to avoid its saturation. These results may match our intuitive expectation.

On the other hand, as indicated by the numerical results in Fig. 3, the critical value of the isolation capacity q_c may have a non-monotonic relation to the value of $1/\sigma_0$ which means the expected time length for the quarantine operation to detect and isolate an infective. We can obtain the following analytical result about the dependence of q_c on $1/\sigma_0$ (Appendix D):

Corollary 5.1.3 *If* $\beta/\gamma \leq 1$, *the critical value of the isolation capacity* q_c *is monotonically decreasing in terms of* $1/\sigma_0$. *On the other hand, there exists a finite value of* $1/\sigma_0$ *to maximize* q_c *if*

$$\frac{\beta}{\gamma} > \frac{s_0 - 1}{s_0 \ln s_0}. \tag{7}$$

It is easily seen that the right side of (7) is greater than 1 for any $s_0 \in (0,1)$.

Sufficiently low efficiency of quarantine operation corresponds to sufficiently large value of $1/\sigma_0$, which means much slow quarantine operation to isolate the infectives in the community. In such a case, the isolated subpopulation size Q increases much slow, so that it is less likely to reach the capacity $Q_{\max}$ on the way of epidemic process. Such a dependence of q_c on $1/\sigma_0$ appears as the decreasing monotonicity of q_c for sufficiently large value of $1/\sigma_0$.

The public health policy must require a high efficiency of quarantine operation. The higher efficiency of quarantine operation leads to a faster increase of Q, and eventually it could become more likely that the isolation reaches the capacity, whereas such an efficient quarantine operation could make the final epidemic size smaller as we will see in the later section.

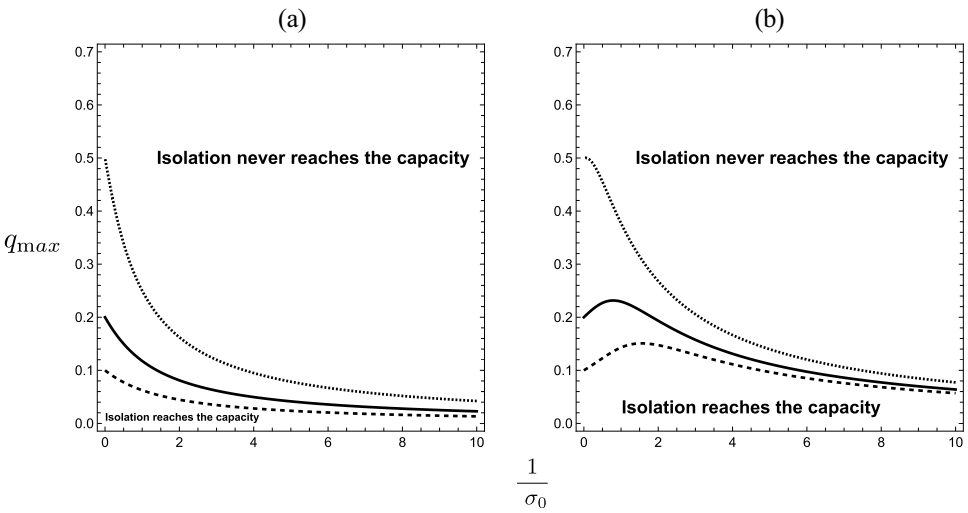

Fig. 3 $(1/\sigma_0)$-dependence of the critical value of the isolation capacity q_c determined by (6) in Corollary 5.1.1. The curves are numerically drawn q_c for $s_0 = 0.5$ (dotted), 0.8 (solid), 0.9 (dashed) with **a** $\beta = 1.0$ and $\gamma = 1.5$; **b** $\beta = 1.5$ and $\gamma = 1.0$. If and only if $q_{\max} < q_c$, the isolation reaches the capacity in a finite time on the way of epidemic process. Refer to Corollary 5.1.3 too. From (6), we can easily find that $q_c \to 1 - s_0$ as $1/\sigma_0 \to +0$

Final size equation

The final epidemic size for system (2) is defined here as the proportion of recovered or isolated individuals in the community at the end of epidemic dynamics. In this section, we show the equation to determine the final epidemic size, respectively, when the isolation never reaches the capacity and when the isolation reaches the capacity on the way of epidemic process, which can be derived from the conserved quantities obtained in Sect. 4.

Final size equation for $q_{\max} \geq q_c$

When the isolation never reaches the capacity in any time, the final epidemic size is determined only by the isolation effective phase. In this case, we can derive the following equation which the final epidemic size $z_\infty^- = q_\infty^- + r_\infty^-$ satisfies (Appendix E):

$$\left(1 - z_\infty^-\right)^{-\sigma_0/\beta}\left(1 + \frac{\gamma}{\sigma_0} - z_\infty^-\right) = \left(s_0\right)^{-\sigma_0/\beta}\left(1 + \frac{\gamma}{\sigma_0}\right). \quad (8)$$

It is proved in Appendix F that equation (8) determines a unique final epidemic size $z_\infty^- \in (1 - s_0, 1)$.

Final size equation for $q_{\max} < q_c$

When the isolation reaches the capacity in a finite time due to its insufficient capacity, we can derive the following equation which the final epidemic size $z_\infty^+ = q_{\max} + r_\infty^+$ satisfies (Appendix E):

$$\frac{\beta}{\sigma_0}\left\{\frac{q_{\max}\left(1 + \sigma_0/\gamma\right)}{1 - q_{\max}} + \ln\left(1 - q_{\max}\right)\right\} = \ln(1 - z_\infty^+) - \ln s_0 + \frac{(\beta/\gamma)z_\infty^+}{1 - q_{\max}}. \quad (9)$$

It is proved in Appendix F that equation (9) determines a unique final epidemic size $z_\infty^+ \in (1 - s(t^\star), 1)$, where $s(t^\star) = (1 - q_{\max})^{\beta/\sigma_0} s_0$ from (E20) in Appendix E, and $1 - s(t^\star) > q_{\max}(1 + \gamma/\sigma_0) > q_{\max}$.

As a result, we have the following theorem:

Theorem 6.1 *The final epidemic size for system* (2) *is uniquely determined by equations* (8) *or* (9) *for given initial condition, which is* $z_\infty^- \in (1 - s_0, 1)$ *for* $q_{\max} \geq q_c$, *and* $z_\infty^+ \in (1 - s(t^\star), 1)$ *for* $q_{\max} < q_c$ *with* $s(t^\star) = (1 - q_{\max})^{\beta/\sigma_0} s_0$.

Dependence of the final epidemic size on $q_{\max}$

The final epidemic size depends on $q_{\max}$ only when the isolation reaches the capacity in a finite time: z_∞^+ depends on $q_{\max}$, while z_∞^- does not. From equation (9), we can find that the final epidemic size z_∞^+ is monotonically decreasing in terms of $q_{\max}$, since $\partial z_\infty^+ / \partial q_{\max}$ is shown to be negative. Figure 4 numerically shows the $q_{\max}$-dependence of the final epidemic size. It is seen that increasing the isolation capacity makes the final epidemic size smaller. This result indicates that the sufficient capacity of isolation could work as an effective factor to suppress a disease spread.

Figure 4b shows a case where the final epidemic size becomes drastically large if the isolation reaches the capacity in a finite time. We obtain the following analytical result on the $q_{\max}$-dependence of the final epidemic size for our model (Appendix G):

Theorem 7.1 *The final epidemic size has a discontinuous change at the critical value of the isolation capacity*: $q_{\max} = q_c$ *such that*

$$z_\infty^\dagger := \lim_{q_{\max} \to q_c - 0} z_\infty^+ > z_\infty^-$$

if and only if

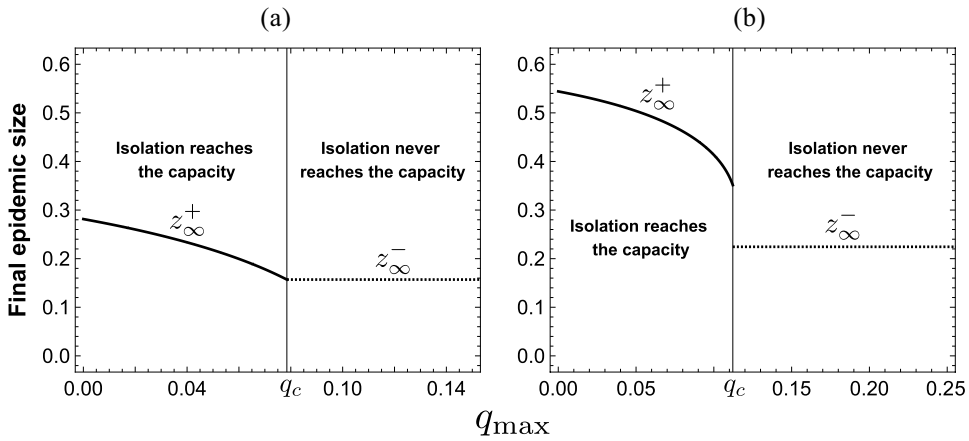

Fig. 4 $q_{\max}$-dependence of the final epidemic size. Numerically drawn for **a** $\beta/\sigma_0 = 0.8$ $(\gamma/\beta = 1.25)$; **b** $\beta/\sigma_0 = 1.25$ $(\gamma/\beta = 0.8)$, $s_0 = 0.9$ and $\gamma/\sigma_0 = 1$

$$\frac{\beta}{\gamma} > 1 \quad \text{and} \quad s_0 > \frac{\gamma}{\beta}\left(1 + \frac{1-\gamma/\beta}{\gamma/\sigma_0}\right)^{\beta/\sigma_0 - 1}. \tag{10}$$

Otherwise it holds that $z_\infty^\dagger = z_\infty^-$.

When condition (10) is not satisfied, the final epidemic size has no discontinuous change at $q_{\max} = q_c$, as numerically illustrated in Fig. 4a. When condition (10) is satisfied, the final epidemic size has a discontinuous change at $q_{\max} = q_c$, as numerically illustrated in Fig. 4b. This means that there is an epidemic situation in which the isolation capacity would be the more important factor for the suppression of disease spread. In such a situation, the insufficiency of isolation capacity could cause a drastically severe consequence of the epidemic dynamics.

Figure 5a shows the parameter region $(\gamma/\sigma_0, \beta/\sigma_0)$ with respect to the discontinuous change of the final epidemic size at $q_{\max} = q_c$. It is seen that for sufficiently small $\beta/\sigma_0 > \gamma/\sigma_0$, such a discontinuous change of the final epidemic size is likely to occur at $q_{\max} = q_c$. It is the case where the disease spread is very slow and the recovery from the disease takes sufficiently long time. Thus the severity of insufficient isolation capacity appears especially for the epidemic dynamics of an infectious disease such that the infectivity is weak, while the disease is hardly treated to the recovery. On the other hand, when $\beta > \gamma$, such a discontinuous change may occur for σ_0 large enough that β/σ_0 becomes sufficiently small. We will see further such a dependence of the final epidemic size on σ_0 in the next section.

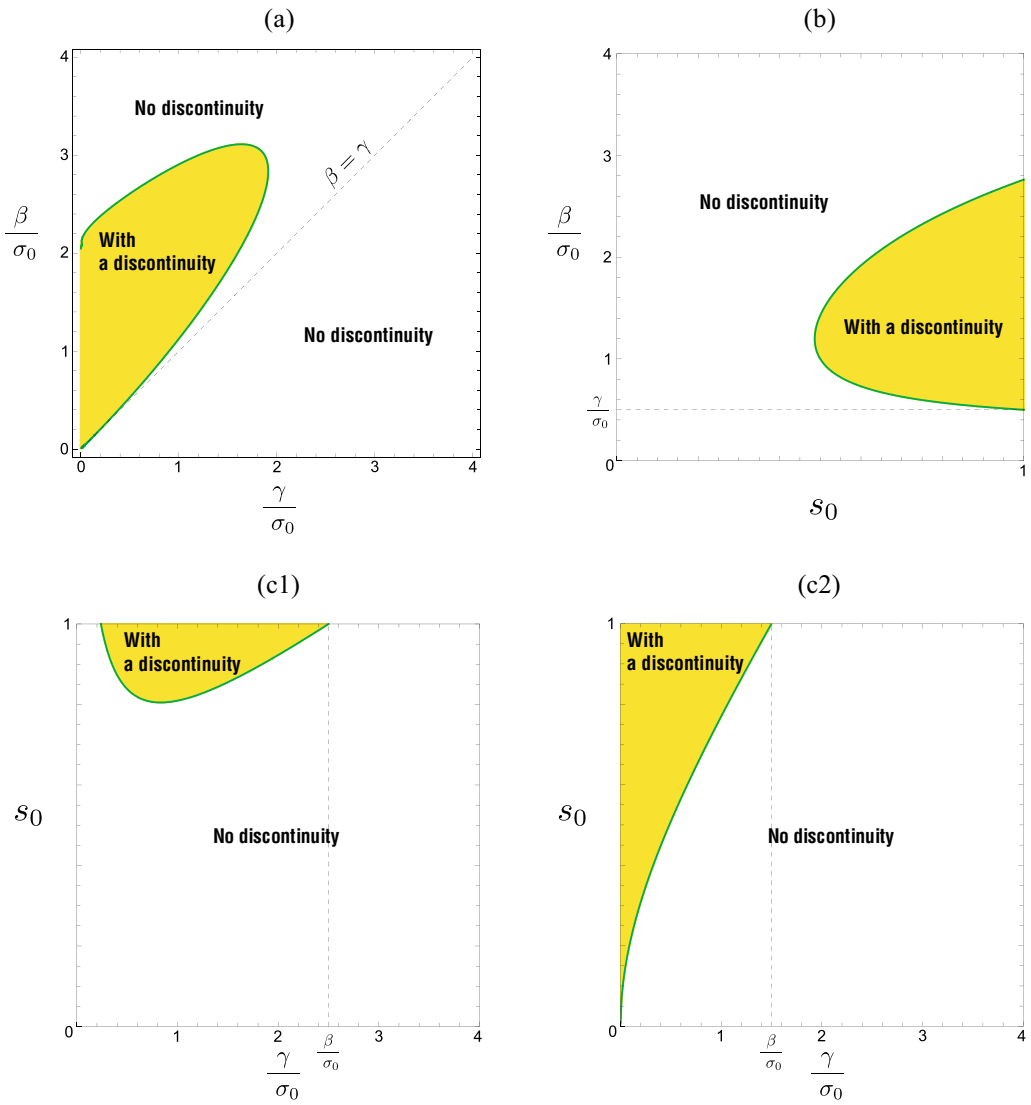

Fig. 5 Parameter region with respect to the discontinuous change of the final epidemic size at $q = q_c$. Numerically drawn with condition (10) in Theorem 7.1 for **a** $s_0 = 0.9$; **b** $\gamma/\sigma_0 = 0.5$; **c1** $\beta/\sigma_0 = 2.5$; **c2** $\beta/\sigma_0 = 1.5$

Figure 5b, c indicates moreover that such a discontinuous change may occur only for sufficiently large s_0, that is, for sufficiently small i_0. This can be regarded as a typical situation as the initial condition about the epidemic dynamics which starts with the invasion of an infectious disease in a community.

Dependence of the final epidemic size on quarantine efficiency

Since the dependence of the critical value of the isolation capacity q_c on the expected time length to detect and isolate an infective $1/\sigma_0$ is not simple as shown by Corollary 5.1.3 in Sect. 5, so is the dependence of the final epidemic size on $1/\sigma_0$. Actually numerical calculations in Fig. 6 indicates such a non-simple dependence of the final epidemic size on $1/\sigma_0$.

As found in Sect. 7, a discontinuous change of the final epidemic size may appear for a specific value of $1/\sigma_0$ (Figs. 6b2, b3). From Theorem 7.1, such a discontinuous change occurs for sufficiently small value of $1/\sigma_0$ only if $\beta > \gamma$. Figure 6b2 shows a numerical example in which there are two critical values of $1/\sigma_0$. Then, a discontinuous change appears only at the smaller critical value of $1/\sigma_0$, while z_∞^+ and z_∞^- continuously connect at the larger critical value of $1/\sigma_0$. In contrast, Fig. 6b3 shows a different example in which such a discontinuous change appears at each of two critical values of $1/\sigma_0$.

For Figs. 6a1, a2, the critical value of the isolation capacity q_c is monotonically decreasing in terms of $1/\sigma_0$ as shown in Fig. 3a, since they are the case where $\beta < \gamma$ (refer to Corollary 5.1.3). Especially for sufficiently small $1/\sigma_0$ and $q_{\max} < 1 - s_0$, the isolation may reach the capacity as seen in Fig. 6a1. Since a sufficiently small $1/\sigma_0$ means a very efficient quarantine operation to detect and isolate the infective sufficiently fast, the isolation is much likely to reach the capacity early. At the isolation incapable phase in such a case, the final epidemic size z_∞^+ becomes larger as the quarantine efficiency gets higher, that is, as the isolation reaches the capacity earlier. These arguments may be applicable also for the case of Fig. 6b1 with a small capacity of isolation.

Numerical results in Fig. 6 show monotonically decreasing of z_∞^+ in terms of $1/\sigma_0$ for every case. In contrast, when the isolation never reaches the capacity, the final epidemic size z_∞^- is monotonically increasing in terms of $1/\sigma_0$, as indicated in Fig. 6. Indeed from Theorem 6.1, we can obtain the following corollary of an analytical result about the $(1/\sigma_0)$-dependence of the final epidemic size (Appendix H):

Corollary 6.1.1 *In terms of* $1/\sigma_0$, *the final epidemic size* z_∞^- *is monotonically increasing, while* z_∞^+ *is monotonically decreasing.*

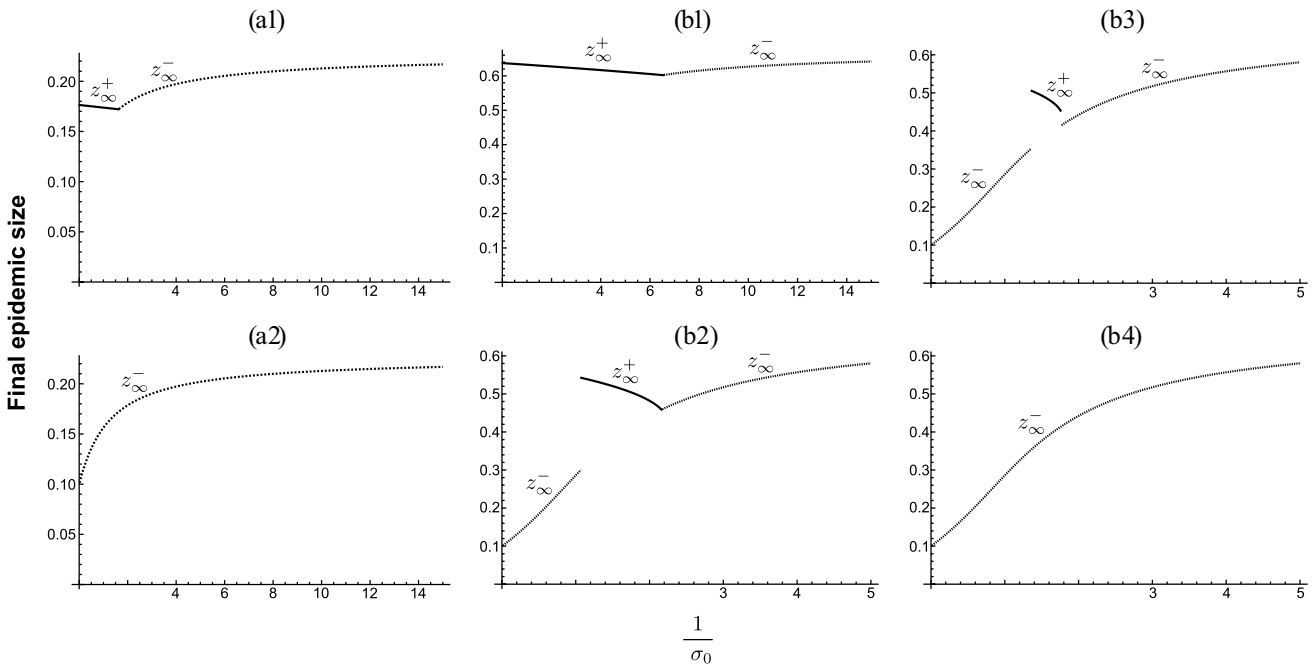

Fig. 6 $(1/\sigma_0)$-dependence of the final epidemic size. Numerically drawn with (8) and (9) for $(\beta, \gamma, q_{\max})$ = **a1** (1.0, 1.5, 0.05); **a2** (1.0, 1.5, 0.10); **b1** (1.5, 1.0, 0.08); **b2** (1.5, 1.0, 0.145); **b3** (1.5, 1.0, 0.15); **b4** (1.5, 1.0, 0.16), and $s_0 = 0.9$. Refer to Theorem 7.1 too

Hence we have some cases like Figs. 6a1, b1 in which there exists a specific value of $1/\sigma_0$ to make the final epidemic size minimum. The specific value of $1/\sigma_0$ must satisfy equation (6) for $q_c = q_{\max}$. In such a case, too efficient quarantine operation makes the final epidemic size larger, due to too early breakdown of the isolation in the epidemic dynamics. However this is not the case for the other situations in Figs. 6a2, b2, b3, b4, when sufficiently high-efficient quarantine operation leads to a smaller final epidemic size.

Mathematically, the case like Figs. 6a1, b1 can appear if and only if condition (5) holds for $1/\sigma_0 \to +0$, since the final epidemic size is given by z_∞^+ for $1/\sigma_0 \to +0$. Thus we find the condition that $q_{\max} < 1 - s_0 = i_0$, as seen for Figs. 6a1, b1. In general, the initial infective subpopulation size i_0 is sufficiently small at the beginning of epidemic dynamics. Hence this could be regarded as the case of a much poor capacity available for the isolation. The above result on the $(1/\sigma_0)$-dependence of the final epidemic size implies that there must be an appropriate management about the quarantine efficiency to suppress the final epidemic size, avoiding the breakdown of isolation.

As seen from Fig. 6, we note that sufficiently low efficiency of the quarantine operation (i.e., sufficiently large $1/\sigma_0$) necessarily leads to a large final epidemic size, even though the isolation never reaches the capacity. Thus the poorly efficient quarantine operation must be improved to make the final epidemic size smaller as mentioned in Sect. 5.

In reality, the efficiency of quarantine operation must depend on the technique, strategy, manpower, and cost available for the public health policy. Therefore it may be difficult to make the efficiency sufficiently high. Further, due to the limited resources available for the public health policy, the efficiency of quarantine operation could have a trade-off relation to the isolation capacity. In conclusion from the results obtained in this section, we remark that the breakdown of isolation must be avoided to make the final epidemic size smaller, taking into account also the efficiency of quarantine operation.

Concluding remarks

In our model with the assumption that any isolated individual is not discharged during the considered epidemic season, the isolation significantly affects the infection force which is determined by the likelihood for a susceptible to contact the pathogen that could have a positive correlation with the density of free infectives in the community. The infection force may become larger under such a permanent isolation because the isolation certainly reduces the number of free members in the community. At the same time, the isolation can play a role to reduce the risk of infection in the community because it certainly decreases the number of free infectives in the community. Our mathematical model clearly indicates that the increase of the isolation capacity makes the final epidemic size smaller, while there would be some unexpected results with such counteracting effects of the isolation on the epidemic dynamics.

It is implied that, once the isolation reaches the capacity and becomes incapable, the final epidemic size may become much large. The occurrence of such a much large final epidemic size depends on the characteristic of epidemic dynamics. When the spread of disease is very slow and the recovery from the disease takes a sufficiently long time, it is likely to occur. Therefore, the severity of insufficient isolation capacity appears especially for the epidemic dynamics of an infectious disease such that the infectivity is weak, while the disease is hardly treated to the recovery. In contrast, for an infectious disease which has a high infectivity or is easily treated to the recovery, the increase in the isolation capacity may be recognized as a relatively small change of the final epidemic size because such a drastic change in the final epidemic size may not be induced by it, even though it certainly works to reduce the final epidemic size.

The smaller critical value of the isolation capacity (q_c in our model) is better for the management of the epidemic dynamics. That is, the smaller critical value for the isolation capacity makes the isolation policy expected to avoid its breakdown with a limited capacity available for it. The larger critical value for the isolation capacity indicates a harder situation for the public health policy since a sufficiently large capacity is necessary to avoid the breakdown of isolation and make the final epidemic size at a low level.

Besides, the analysis of our model gives the results on the importance of quarantine operation in order to make the isolation more effective to induce the smaller epidemic size. For a limited capacity of isolation, too high efficiency of the quarantine operation would be inappropriate for the purpose to suppress the epidemic size. This result implies that it is not easy to design the isolation strategy optimal, and thus, for example, a satisfactory monitoring system to appropriately carry out the isolation policy is required to avoid the breakdown of isolation.

The results from our model imply that the breakdown of isolation with a limited capacity would cause a drastic increase of the epidemic size. An insufficient capacity of the isolation could lead to an unexpectedly severe epidemic situation. In order words, a sufficiently large capacity of the isolation could serve the suppression of epidemic size enough. However such an efficiency of the large isolation capacity would be supplementary, depending on the nature of the epidemic dynamics. Therefore, the isolation could not be necessarily the principal factor for the public health against the spread of an infectious disease, while it must be important and could have a significant contribution to make

the epidemic size smaller, naturally accompanied with the other policies against the epidemics.

Our theoretical results would highlight the importance of satisfactory infrastructure for the public health as indicated by Unruh *et al.* (2022) on the social response to the COVID-19 pandemic. Since the satisfactory infrastructure for the public health needs a sufficient social investment, those arguments on our model would imply a difficulty of the management of even isolation policy against an infectious disease spreading in a community too.

Appendix A Meaning of parameter β for $\mathscr{R}_0$

In our model (1), the increase of infective population size in a sufficiently short interval $[t, t+\Delta t]$ is given by

$$\beta \frac{I(t)}{N-Q(t)} S(t)\Delta t + \mathrm{o}(\Delta t) = \beta \frac{I(t)}{S(t)+I(t)+R(t)} S(t)\Delta t + \mathrm{o}(\Delta t).$$

Hence the expected number of new cases produced by an infective individual in $[t, t+\Delta t]$ becomes

$$\beta \frac{S(t)}{S(t)+I(t)+R(t)} \Delta t + \mathrm{o}(\Delta t). \tag{A1}$$

Since the basic reproduction number is defined as such an expected number of new cases when the infective individual contacts only susceptibles, it is regarded as the epidemiological index in an ideal situation for the production of new cases by the infective. In this sense, we can consider the supremum of the expected number of new cases in the ideal situation for the epidemic dynamics to derive the basic reproduction number (Seno 2022). For this reason, to derive the basic reproduction number $\mathscr{R}_0$ for our model (1), it is sufficient for us to consider only the epidemic dynamics at the isolation effective phase, since the isolation incapable phase corresponds to a non-ideal situation in which the number of susceptible individuals has become small. Above formula (A1) is monotonically increasing in terms of $S(t)$, while sufficiently large $S(t)$ makes $I(t)+R(t)$ small because of $S+I+Q+R=1$. Therefore the supremum of (A1) can be given for $(S,I,R)=(N,0,0)$, which makes it $\beta\Delta t + \mathrm{o}(\Delta t)$. This result indicates that β in our model (1) *corresponds* to the supremum of the expected number of new cases produced by an infective per unit time. Remark that, in the actual epidemic dynamics even with our model (and generally with any other model), any temporal change in the other subpopulation sizes makes the expected number of new cases produced by an infective smaller, that is, the expected number of new cases per unit time temporally changes necessarily below the supremum.

Furthermore, the mathematically obtained basic reproduction number in general is given by a formula expressing the product of such a supremum of the expected number of new cases per unit time and the expected duration of infectivity for an infective. Also from this viewpoint, β in our model can have the meaning of the supremum of the expected number of new cases produced by an infective per unit time, because the expected duration of infectivity for an infective at the isolation effective phase is given by $1/(\gamma+\sigma_0)$ for our model (1).

Appendix B Derivation of the conserved quantity at each phase

The isolation effective phase

When the isolation never reaches the capacity in a finite time due to a sufficient isolation capacity for the epidemic dynamics, system (2) always follows the isolation effective phase with $\hat{\sigma}(q)=\hat{\sigma}_0$. In this case, from the equations in (2), we can derive the following differential equations:

$$\frac{di}{ds} = -1 + \frac{1-q}{\mathscr{R}_0 s}, \tag{B2}$$

where we used the equality $\hat{\sigma}_0 + \hat{\gamma} = 1$, and

$$\frac{dq}{dr} = \frac{\hat{\sigma}_0}{\hat{\gamma}}. \tag{B3}$$

From (B3), we can obtain the following relation between q and r:

$$q = \frac{\sigma_0}{\gamma} r, \tag{B4}$$

where we used $q(0)=r(0)=0$. Since $s+i+q+r=1$, equation (B4) becomes

$$1-q = \frac{s+i+\gamma/\sigma_0}{1+\gamma/\sigma_0}. \tag{B5}$$

Substituting (B5) for (B2), we can derive the following ordinary differential equation of i in terms of s:

$$\frac{d}{ds}\left(i s^{-\sigma_0/\beta}\right) = s^{-\sigma_0/\beta}\left(-1+\frac{1}{\beta/\sigma_0}\right) + \frac{s^{-1-(\sigma_0/\beta)}}{\beta/\gamma}.$$

We can easily solve this ordinary differential equation, and get relation (3), making use of $i(0)=i_0$, $s(0)=s_0$, and $i_0+s_0=1$.

The isolation incapable phase

Once the isolation reaches the capacity in a finite time on the way of the epidemic process with an insufficient isolation capacity, system (2) comes to follow the isolation incapable

phase. In this case, from the first and second equations of (2), we can derive the following differential equation:

$$\frac{di}{ds} = -1 + \frac{1 - q_{\max}}{\mathscr{R}_0\left(1 + \sigma_0/\gamma\right)s}. \tag{B6}$$

We can easily solve (B6) and get the relation

$$i(\hat{t}) = -s(\hat{t}) + \frac{\gamma}{\beta}(1 - q_{\max}) \ln s(\hat{t}) + C \tag{B7}$$

with an undetermined constant C. For $\hat{t} = t^\star$, we have

$$C = s(t^\star) + i(t^\star) - \frac{\gamma}{\beta}(1 - q_{\max}) \ln s(t^\star). \tag{B8}$$

Making use of (B8) for (B7), we can get equation (4) that gives the conserved quantity at the isolation incapable phase.

Appendix C Proof for theorem 5.1 and Corollaries 5.1.1 and 5.1.2

From equation (3), when the isolation never reaches the capacity, we have the equation

$$s_\infty^- = F(s_\infty^-) := -\frac{\gamma}{\sigma_0} + \left(\frac{s_\infty^-}{s_0}\right)^{\sigma_0/\beta}\left(1 + \frac{\gamma}{\sigma_0}\right), \tag{C9}$$

since $i(\hat{t}) \to 0$ as $\hat{t} \to \infty$. We have $F(0) = -\gamma/\sigma_0$, $F(s_0) = 1$, and $F'(s) > 0$ for $s \in (0, s_0)$. Hence $F(s)$ is a monotonically increasing, continuous and differentiable function of $s \in (0, s_0)$, which satisfies that $F(0) < 0$ and $F(s_0) > s_0$. Further we can easily find that $F(s)$ is linear if $\sigma_0/\beta = 1$, and otherwise it is alternatively convex or concave for $s \in (0, s_0)$. Therefore we find that equation (C9) has a unique root $s_\infty^- \in (0, s_0)$, and $F(s) < s$ for $s \in (0, s_\infty^-)$, while $F(s) > s$ for $s \in (s_\infty^-, s_0)$.

On the other hand, from $s_\infty^- + q_\infty^- + r_\infty^- = 1$ and $q_\infty^- = (\sigma_0/\gamma) r_\infty^-$ by (B4) in Appendix B, we find that $s_\infty^- = 1 - (1 + \gamma/\sigma_0) q_\infty^-$. Making use of this relation, we find that equation (C9) is equivalent to the following equation:

$$q_\infty^- = 1 - \left(\frac{s_\infty^-}{s_0}\right)^{\sigma_0/\beta}. \tag{C10}$$

It must be satisfied that $q_\infty^- \le q_{\max}$ in the case where $q(\hat{t})$ never reaches $q_{\max}$ for any $\hat{t} > 0$. Since $q(\hat{t})$ is monotonically increasing in terms of $\hat{t}$, if $q_\infty^- \le q_{\max}$, the isolation does not reach the capacity for any $\hat{t} > 0$. Therefore, if and only if $q_\infty^- \le q_{\max}$, the isolation does not reach the capacity for any $\hat{t} > 0$.

Consequently we find that, if and only if $q_\infty^- > q_{\max}$, the isolation reaches the capacity at $\hat{t} = t^\star < \infty$. From (C9) and (C10), we can derive the following condition equivalent to $q_\infty^- > q_{\max}$:

$$s_\infty^- < 1 - q_{\max}\left(1 + \frac{\gamma}{\sigma_0}\right). \tag{C11}$$

Since $s_\infty^- > 0$, we note that this inequality holds only if $q_{\max} < 1/(1 + \gamma/\sigma_0)$. Hence if $q_{\max} \ge 1/(1 + \gamma/\sigma_0)$, inequality (C11) does not hold and we necessarily have $q_\infty^- \le q_{\max}$, so that the isolation does not reach the capacity for any $\hat{t} > 0$. From the nature of the function $F(s)$ shown in the above, condition (C11) is equivalent to the condition that $F(s) > s$ for $s = 1 - q_{\max}(1 + \gamma/\sigma_0)$. This leads to condition (5) in Theorem 5.1.

On the other hand, from condition (C11), we can define the critical value for the isolation capacity q_c as $q_c := (1 - s_\infty^-)/(1 + \gamma/\sigma_0)$ such that condition (C11) is satisfied if and only if $q_{\max} < q_c$, which becomes necessary and sufficient for the isolation to reach the capacity in a finite time. Substituting $s_\infty^- = 1 - q_c(1 + \gamma/\sigma_0)$ for the equation $F(s_\infty^-) = s_\infty^-$, we can get equation (6). Then the uniqueness of q_c follows that of s_∞^- shown in the above.

Appendix D Proof for corollary 5.1.3

From (6), we can easily derive the following derivative of q_c in terms of $1/\sigma_0$:

$$\frac{\partial q_c}{\partial(1/\sigma_0)} = \frac{\sigma_0^2 \gamma q_c + \sigma_0 \beta\left\{\sigma_0 - q_c\left(\sigma_0 + \gamma\right)\right\} \ln(1 - q_c)}{-\sigma_0\left(\sigma_0 + \gamma\right) + \beta\left\{\sigma_0 - q_c\left(\sigma_0 + \gamma\right)\right\}/(1 - q_c)}. \tag{D12}$$

As we can easily find from (6) that $q_c \to 1 - s_0$ as $1/\sigma_0 \to +0$, we have

$$\left.\frac{\partial q_c}{\partial\left(1/\sigma_0\right)}\right|_{(1/\sigma_0, q_c) \to (+0, 1 - s_0)} > 0 \iff \frac{\beta}{\gamma} > \frac{s_0 - 1}{s_0 \ln s_0}. \tag{D13}$$

Next, to find the sign of (D12) for sufficiently large value of $1/\sigma_0$, we use the Maclaurin expansion in terms of σ_0 and get

$$\frac{\partial q_c}{\partial\left(1/\sigma_0\right)} = (1 - q_c) \ln(1 - q_c)\sigma_0 + \mathrm{o}\left(\sigma_0\right).$$

Since $(1 - q_c) \ln(1 - q_c) < 0$ for $q_c \in (0, 1)$, the sign of (D12) must be necessarily negative for sufficiently large value of $1/\sigma_0$. As a consequence, q_c is monotonically decreasing for sufficiently large value of $1/\sigma_0$.

Since q_c is continuous in terms of $1/\sigma_0$, q_c is monotonically increasing for a sufficiently small value of $1/\sigma_0$ if condition (D13) is satisfied. Then, q_c has at least one extremal maximum for a finite value of $1/\sigma_0$. It is easily seen that $(s_0 - 1)/(s_0 \ln s_0) > 1$ for any $s_0 \in (0, 1)$.

On the other hand, the following equation must be satisfied at the extremum that makes derivative (D12) zero:

$$1+\frac{\gamma}{\sigma_0}=\frac{1}{q_c}+\frac{\gamma/\beta}{\ln(1-q_c)}. \tag{D14}$$

We can easily prove that the right side of (D14) is less than 1 for any $q_c \in (0,1)$ if $\beta/\gamma \le 1$. Since the left side of (D14) is always greater than 1, this means that equation (D14) cannot hold for $\beta/\gamma \le 1$. Hence, derivative (D12) cannot become zero if $\beta/\gamma \le 1$.

Therefore, $\beta/\gamma > 1$ is necessary for the existence of a certain value of $1/\sigma_0 > 0$ to maximize q_c. At the same time, this result means that, when $\beta/\gamma \le 1$, derivative (D12) cannot change the sign at any value of $1/\sigma_0$. Then from the above arguments, it must be negative, so that q_c is monotonically decreasing in terms of $1/\sigma_0$ when $\beta/\gamma \le 1$.

Appendix E Derivation of the final size equation

Final size equation for $q_{\max} \ge q_c$

By applying $\hat{t} \to \infty$ for equation (3), we get the following equation:

$$(s_\infty^-)^{-\sigma_0/\beta}\left(s_\infty^- + \frac{\gamma}{\sigma_0}\right) = (s_0)^{-\sigma_0/\beta}\left(1+\frac{\gamma}{\sigma_0}\right), \tag{E15}$$

where we used $i(\hat{t}) \to 0$ as $\hat{t} \to \infty$. The final epidemic size is given by $z_\infty^- = q_\infty^- + r_\infty^- = 1 - s_\infty^-$. Making use of $s_\infty^- = 1 - z_\infty^-$ for (E15), we can get equation (8) which determines the final epidemic size when the isolation never reaches the capacity.

Final size equation for $q_{\max} < q_c$

By applying $\hat{t} \to \infty$ for equation (4), we can get the following equation:

$$s_\infty^+ = s(t^\star) + i(t^\star) + \frac{\gamma}{\beta}(1-q_{\max})\ln\frac{s_\infty^+}{s(t^\star)}, \tag{E16}$$

where we used $i(\hat{t}) \to 0$ as $\hat{t} \to \infty$. Now, from the equality $s(\hat{t}) + i(\hat{t}) = 1 - \{q(\hat{t}) + r(\hat{t})\}$ and (B4) derived in Appendix B, we have

$$s(\hat{t}) + i(\hat{t}) = 1 - q(\hat{t})\left(1+\frac{\gamma}{\sigma_0}\right). \tag{E17}$$

For the continuity of the solution at $\hat{t} = t^\star$, we have $s(\hat{t}) = s(t^\star)$, $i(\hat{t}) = i(t^\star)$, and $q(t^\star) = q_{\max}$. Then equations (3) and (E17) become

$$s(t^\star) + i(t^\star) = -\frac{\gamma}{\sigma_0} + \left\{\frac{s(t^\star)}{s_0}\right\}^{\sigma_0/\beta}\left(1+\frac{\gamma}{\sigma_0}\right); \tag{E18}$$

$$s(t^\star) + i(t^\star) = 1 - q_{\max}\left(1+\frac{\gamma}{\sigma_0}\right). \tag{E19}$$

We can solve parallel equations (E18) and (E19) in terms of $s(t^\star)$ and $i(t^\star)$,

$$s(t^\star) = (1-q_{\max})^{\beta/\sigma_0}s_0; \quad i(t^\star) = 1 - s(t^\star) - q_{\max}\left(1+\frac{\gamma}{\sigma_0}\right), \tag{E20}$$

and then substitute them for (E16). As a result, we can get the equation

$$\begin{aligned}&\frac{\beta/\gamma}{1-q_{\max}}\left\{s_\infty^+ - 1 + q_{\max}\left(1+\frac{\gamma}{\sigma_0}\right)\right\}\\ &\quad = \ln s_\infty^+ - \ln s_0 - \frac{\beta}{\sigma_0}\ln(1-q_{\max}).\end{aligned} \tag{E21}$$

When the isolation reaches the capacity at any finite time, the final epidemic size is defined by $z_\infty^+ = q_{\max} + r_\infty^+ = 1 - s_\infty^+$. Thus, making use of $s_\infty^+ = 1 - z_\infty^+$ for (E21), we can get equation (9).

Appendix F Proof for the unique existence of the final epidemic size

Unique existence of z_∞^-

The left side of equation (8) is a function of z_∞^-, which we denote here by $A(z_\infty^-)$. The right side of (8) is a positive constant B_0 independent of z_∞^-. The function $A(z)$ is continuous and differentiable for $z \in (1-s_0, 1)$, satisfying that

$$A(1-s_0) = (s_0)^{-\sigma_0/\beta}\left(s_0 + \frac{\gamma}{\sigma_0}\right) < B_0; \quad \lim_{z\to 1-0} A(z) = \infty > B_0.$$

Hence, there exists at least one root of the equation $A(z) = B_0$ for $z \in (1-s_0, 1)$.

We can easily find that the function $A(z)$ is monotonically increasing or has a unique extremal minimum in $(1-s_0, 1)$. When $A(z)$ is monotonically increasing for $z \in (1-s_0, 1)$, it must have a unique intersection with the horizontal line B_0 in $(1-s_0, 1)$. Even when $A(z)$ has a unique extremal minimum for $z \in (1-s_0, 1)$, it has a unique intersection with the horizontal line B_0 since $A(1-s_0) < B_0$. Thus in both cases, the equation $A(z) = B_0$ has a unique root in $(1-s_0, 1)$. As a result, the final epidemic size $z_\infty^- \in (1-s_0, 1)$ is uniquely determined by equation (8).

Unique existence of z_∞^+

To prove that the final epidemic size z_∞^+ is uniquely determined by equation (9), let us consider the existence of a root for the equation $G(s) = 0$ where

$$G(s) := s - \left\{s(t^\star) + i(t^\star)\right\} - \frac{\gamma}{\beta}(1 - q_{\max}) \ln \frac{s}{s(t^\star)}. \quad \text{(F22)}$$

From (4) in Sect. 4 and (E16) in Appendix E, the equation $G(1 - z_\infty^+) = 0$ is mathematically equivalent to final size equation (9). Since $s(\hat{t})$ is monotonically decreasing as time passes, we have $s(\hat{t}) < s(t^\star)$ for $\hat{t} > t^\star$. Hence we consider $G(s)$ hereafter for $s \in (0, s(t^\star))$. The function $G(s)$ is continuous and differentiable for $s \in (0, s(t^\star))$. Moreover, it satisfies that $\lim_{s \to +0} G(s) = \infty > 0$, and $G(s(t^\star)) = -i(t^*) < 0$. From these facts, the equation $G(s) = 0$ has at least one root in $(0, s(t^\star))$. Further we can easily find that $G(s)$ is monotonically decreasing or has a unique extremal minimum in $(0, s(t^\star))$. When $G(s)$ is monotonically decreasing for $s \in (0, s(t^\star))$, the equation $G(s) = 0$ has a unique root in $(0, s(t^\star))$. Even when $G(s)$ has a unique extremal minimum in $(0, s(t^\star))$, it has a unique root in $(0, s(t^\star))$, because the extremum value of G must be negative since $G(s(t^\star)) < 0$. Hence in both cases, the equation $G(s) = 0$ has a unique root $s = s_\infty^+ \in (0, s(t^\star))$. Therefore, equation (9) determines a unique final epidemic size $z_\infty^+ \in (1 - s(t^\star), 1)$. This is because $s_\infty^+ = 1 - z_\infty^+$ and $z_\infty^+ = 1 - s_\infty^+ \in (1 - s(t^\star), 1)$ where $1 - s(t^\star) = i(t^\star) + q_{\max}(1 + \gamma/\sigma_0) > q_{\max}(1 + \gamma/\sigma_0) > q_{\max}$ from (E19), and $s(t^\star)$ is given by (E20) in Appendix E.

Appendix G Proof for Theorem 7.1

In order to prove Theorem 7.1, we use two lemmas.

Lemma G.1 *It holds that* $z_\infty^+ \geq q_c\left(1 + \gamma/\sigma_0\right) \geq z_\infty^-$.

Proof The proof is given straightforward from the arguments in the proof for Theorem 5.1 and its corollaries, given in Appendix C. From (C11), the condition $q_\infty^- \leq q_{\max}$ is equivalent to

$$s_\infty^- \geq 1 - q_{\max}\left(1 + \frac{\gamma}{\sigma_0}\right), \quad \text{(G23)}$$

where s_∞^- is the root of (E15), and subsequently q_∞^- is given by (C10). Thus, when and only when condition (G23) is satisfied, the isolation never reaches the capacity, so that the epidemic dynamics is always at the isolation effective phase. Inversely, when and only when condition (G23) is unsatisfied, the epidemic dynamics enters in the isolation incapable phase in a finite time.

Thus, for the value $s(t^\star)$ at the moment when the isolation incapable phase begins, it must hold that

$$s(t^\star) < 1 - q_{\max}\left(1 + \frac{\gamma}{\sigma_0}\right).$$

The value $s(\hat{t})$ is monotonically decreasing in terms of time since $ds/d\hat{t}$ is negative for any $\hat{t} > 0$. Hence we have $s_\infty^+ < s(t^\star)$ where s_∞^+ is the root of (E21) at the isolation incapable phase. Therefore, we have

$$s_\infty^+ < 1 - q_{\max}\left(1 + \frac{\gamma}{\sigma_0}\right). \quad \text{(G24)}$$

Since $z_\infty^- = 1 - s_\infty^-$, these arguments indicate that, when and only when the isolation never reaches the capacity, we have $z_\infty^- \leq q_{\max}(1 + \gamma/\sigma_0)$ from (G23). Since this condition must hold for any $q_{\max} \geq q_c$ from Corollary 5.1.1, and since z_∞^- is independent of $q_{\max}$, we find that $z_\infty^- \leq q_c(1 + \gamma/\sigma_0)$.

On the other hand, when the isolation reaches the capacity at a finite time with $q_{\max} < q_c$, we have $z_\infty^+ > q_{\max}(1 + \gamma/\sigma_0)$ from (G24). Since this condition must hold for any $q_{\max} < q_c$, we have $z_\infty^+ \geq q_c(1 + \gamma/\sigma_0)$. □

Lemma G.2 *It holds that* $z_\infty^- = q_c\left(1 + \gamma/\sigma_0\right)$.

Proof Substituting $z_\infty^- = q_c\left(1 + \gamma/\sigma_0\right)$ for (8) and taking account of (6) in Corollary 5.1.1, we can easily find that equation (E21) holds. Since z_∞^- is uniquely determined as the root of (8), we can result in this lemma. □

Now, equation (9) can be rewritten as

$$q_{\max}\left(1 + \frac{\gamma}{\sigma_0}\right) - z_\infty^+ = \frac{\gamma}{\beta}(1 - q_{\max}) \ln \frac{1 - z_\infty^+}{s_0(1 - q_{\max})^{\beta/\sigma_0}}. \quad \text{(G25)}$$

Taking the limit as $q_{\max} \to q_c$ for (G25), we have the following equation with respect to $z_\infty^\dagger$ from Lemma G.2 and (8):

$$H(z_\infty^\dagger) := z_\infty^- - z_\infty^\dagger - \frac{\gamma}{\beta}(1 - q_c) \ln \frac{1 - z_\infty^\dagger}{1 - z_\infty^-} = 0.$$

It is easily found that $H(z_\infty^-) = 0$ and $\lim_{z \to 1-0} H(z) = \infty$. The derivative of $H(z)$ becomes

$$H'(z) = -1 + \frac{\gamma}{\beta} \frac{1 - q_c}{1 - z},$$

which is monotonically increasing in terms of $z \in (z_\infty^-, 1) \subset (0, 1)$ with $H'(z) \to \infty$ as $z \to 1 - 0$. If $H'(z_\infty^-) \geq 0$, then $H(z) > 0$ for any $z \in (z_\infty^-, 1)$. In this case, the root of $H(z) = 0$ in $[z_\infty^-, 1]$ is only $z = z_\infty^-$. In contrast, if $H'(z_\infty^-) < 0$, there exists a unique value $\eta \in (z_\infty^-, 1)$ such that $H'(z) < 0$ for $z \in (z_\infty^-, \eta)$ and $H'(z) > 0$ for $z \in (\eta, 1)$. This means that $H(z) < 0$ for $z \in (z_\infty^-, \eta)$, and $H(z)$ is monotonically increasing for $z \in (\eta, 1)$ with $\lim_{z \to 1-0} H(z) = \infty$. Therefore we have a unique value $\zeta \in (\eta, 1) \subset (z_\infty^-, 1)$ such that $H(\zeta) = 0$, because $H(z)$ is continuous in $(z_\infty^-, 1)$.

On the other hand, from (9), we can derive

$$\frac{\partial z_\infty^+}{\partial q_{\max}} = \frac{1+(\gamma/\beta)\ln\left[(1-z_\infty^+)/\left\{s_0(1-q_{\max})^{\beta/\sigma_0}\right\}\right]}{1-(\gamma/\beta)(1-q_{\max})/(1-z_\infty^+)}.$$

Then we have

$$\left.\frac{\partial z_\infty^+}{\partial q_{\max}}\right|_{(q_{\max},z_\infty^+)=(q_c,z_\infty^-)} = \frac{1}{1-(\gamma/\beta)(1-q_c)/(1-z_\infty^-)} = -\frac{1}{H'(z_\infty^-)}. \tag{G26}$$

Hence we find that, if $H'(z_\infty^-)<0$, derivative (G26) becomes positive. Thus, if $z_\infty^\dagger = z_\infty^+$ with $H'(z_\infty^-)<0$, z_∞^+ must be smaller than z_∞^- for $q_{\max}$ less than and sufficiently near q_c because z_∞^+ is continuous and differentiable for $q_{\max}\in(0,q_c)$ and derivative (G26) is positive. This is contradictory to the result of Lemma G.1. Therefore, if $H'(z_\infty^-)<0$, $z_\infty^\dagger$ must be greater than z_∞^-.

The condition $H'(z_\infty^-)<0$ is equivalent to the following:

$$\frac{\gamma}{\beta}<1 \ \text{ and } \ q_c<q_{cc}:=\frac{1-\gamma/\beta}{1-\gamma/\beta+\gamma/\sigma_0}. \tag{G27}$$

From $q_{\max}<q_c$ and (6), the second inequality of (G27) is equivalent to

$$1-q_{cc}\left(1+\frac{\gamma}{\sigma_0}\right)<s_0\left(1-q_{cc}\right)^{\beta/\sigma_0}.$$

This inequality results in the second condition of (10). If $H'(z_\infty^-)\geq 0$, $z_\infty^\dagger$ must be z_∞^- since the equation $H(z)=0$ has the unique root $z=z_\infty^-$ in $[z_\infty^-,1]$ and derivative (G26) is non-positive with no contradiction. These arguments prove the theorem.

Appendix H Proof for Corollary 6.1.1

$(1/\sigma_0)$-dependence of z_∞^-

Equation (8) to determine z_∞^- can be rewritten as

$$\ln\frac{1-z_\infty^-}{s_0}=U(z_\infty^-):=\frac{\beta}{\sigma_0}\ln\left(1-\frac{z_\infty^-}{1+\gamma/\sigma_0}\right). \tag{H28}$$

According to the function $U(z)$ for $z\in(1-s_0,1)$, we can easily find that $U(z)$ is monotonically decreasing in terms of $z\in(1-s_0,1)$, and so is the left-hand function of equation (H28), $\ln\{(1-z)/s_0\}$. As already shown in Theorem 6.1, $U(z)$ and $\ln\{(1-z)/s_0\}$ necessarily have a unique intersection in $(1-s_0,1)$, which gives z_∞^-. Since $U(1-s_0)<0$ and $|U(1)|<\infty$, it is satisfied that $U(z)>\ln\{(1-z)/s_0\}$ for $z\in(1-s_0,z_\infty^-)$ and $U(z)<\ln\{(1-z)/s_0\}$ for $z\in(z_\infty^-,1)$.

On the other hand, we can easily show that

$$\lim_{1/\sigma_0\to+0}\frac{\partial U(z)}{\partial(1/\sigma_0)}=\beta\ln(1-z)<0; \quad \lim_{1/\sigma_0\to\infty}\frac{\partial U(z)}{\partial(1/\sigma_0)}=0;$$
$$\frac{\partial^2 U(z)}{\partial(1/\sigma_0)^2}>0.$$

Hence $\partial U(z)/\partial(1/\sigma_0)<0$ for $1/\sigma_0>0$, that is, the value of $U(z)$ is monotonically decreasing in terms of $1/\sigma_0$ for $z\in(1-s_0,1)$. Therefore, the intersection of $U(z)$ and $\ln\{(1-z)/s_0\}$ monotonically decreasing in $(1-s_0,1)$ must move toward the larger z as $1/\sigma_0$ gets larger. This means that z_∞^- is monotonically increasing in terms of $1/\sigma_0$.

$(1/\sigma_0)$-dependence of z_∞^+

Equation (9) to determine z_∞^+ can be rewritten as

$$\frac{\beta K}{\sigma_0}=W(z_\infty^+):=\ln\frac{1-z_\infty^+}{s_0}+\frac{\beta}{\gamma}\frac{z_\infty^+-q_{\max}}{1-q_{\max}}, \tag{H29}$$

where $K:=q_{\max}/(1-q_{\max})+\ln(1-q_{\max})>0$ for $q_{\max}\in(0,1)$. We can easily find that

$$W(1-s(t^\star))=\ln\frac{s(t^\star)}{s_0}+\frac{\beta}{\gamma}\frac{1-s(t^\star)-q_{\max}}{1-q_{\max}} > \frac{\beta}{\sigma_0}\ln(1-q_{\max})+\frac{\beta}{\gamma}\frac{(\gamma/\sigma_0)q_{\max}}{1-q_{\max}}=\frac{\beta K}{\sigma_0}>0,$$

making use of $s(t^\star)=(1-q_{\max})^{\beta/\sigma_0}s_0$ and the inequality that $1-s(t^\star)>q_{\max}(1+\gamma/\sigma_0)$ which was shown at the end of Appendix F. Further it can be easily proved that $W(z)$ is monotonically decreasing in terms of $z\in(1-s(t^\star),1)$ or alternatively has a unique extremal maximum in $(1-s(t^\star),1)$, with $\lim_{z\to1-0}W(z)=-\infty$.

On the other hand, we find that condition (5) in Theorem 5.1 is mathematically equivalent to that $(\beta K/\sigma_0)<W(1-s(t^\star))$. This is mathematically consistent with the definition of z_∞^+ which can exist only if condition (5) is satisfied when the isolation reaches the capacity in a finite time. Hence we can hereafter consider only $1/\sigma_0$ such that $(\beta K/\sigma_0)<W(1-s(t^\star))$, since equation (9) is valid only under condition (5).

Then, from the nature of $W(z)$ shown in the above, equation (H29) has a unique root $z_\infty^+\in(1-s(t^\star),1)$, that is, a unique intersection of $W(z)$ and the horizontal line $\beta K/\sigma_0$ in $(1-s(t^\star),1)$. When $W(z)$ is monotonically decreasing in terms of $z\in(1-s(t^\star),1)$, the intersection moves toward the larger z as $1/\sigma_0$ gets smaller. Even when $W(z)$ has a unique extremal maximum in $(1-s(t^\star),1)$, the intersection giving $z_\infty^+\in(1-s(t^\star),1)$ must be on the decreasing part of curve $W(z)$, since $W(1-s(t^\star))>0$ and $(\beta K/\sigma_0)<W(1-s(t^\star))$. Hence, even in such a case, the intersection moves toward the larger z as $1/\sigma_0$ gets smaller.

Consequently, these arguments prove that z_∞^+ is monotonically decreasing in terms of $1/\sigma_0$.

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