Afrika Matematika (2024) 35:2
https://doi.org/10.1007/s13370-023-01134-9

A rejoinder model for the population dynamics of the spread of two interacting pieces of information

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Received: 20 April 2020 / Accepted: 19 October 2023

Abstract
Information warfare requires more attention as competing interests get escalated by the spread of information on various Internet-based social media platforms in recent times. In this work, we construct and analyze a mathematical model with a system of ordinary differential equations to consider how two interacting pieces of information, where the first one is incomplete and misleading while the second one is corrective of the first, evolve with time in an online population. The counter and correctional information is the rejoinder. Human psychological and sociological attributes like disbelief in the rejoinder and increased tendency to keep spreading the misleading information even after knowing the correct one are factored into our model. We find that in correcting a misleading piece of information that is already spreading within a population, the rejoinder has to be released early enough within a certain time range. The findings can help us appreciate the impact of misinformation on the society and promote information literacy at optimal cost.

Keywords Information transmission · Population dynamics · Mathematical modeling · Rejoinder model · Human attributes · Information literacy

Mathematics Subject Classification 34C60 · 92D25 · 93A30

1 Introduction

The whole existence of nature is built on information and its transmission; in fact, things hardly happen in the universe exclusive of the concepts of information and communication [21, 54, 55]. Parker [50] gives one of the most fundamental definitions of information as 'the pattern of organization of matter and energy'. In a sense, Bates [3] sees information as something which exists objectively in our cosmos but is however handled subjectively by

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Published online: 24 November 2023

individuals in the process of construction, storage and acting upon it. Insights into the nature and attributes of information has been brought to the forefront by a number of authors like [17, 22, 23, 30, 58]. More so, some work has been done on information disorders like rumors, gossips, fake news, etc. (see [5, 12, 28, 29, 32, 47, 53, 63]). The spread of false information through social networks looks so much like the progression of communicable infections since they are both enhanced by social connections. As such, analyzing the dynamics of online propagation and competition of different pieces of information can be quite revealing. With the growing interest among people to source for news on social media primarily, transmission models together with relevant data might be of great help in understanding this new terrain (see [35, 62]).

In a world of rumors, gossips, urban legends, political propaganda and commercial advertisements, there are always loads of information competing for human attention. Many a time, the nature of some information can be quite divisive and get people polarized as they hold on tenaciously to their respective view points. As such, mathematical modelers have been interested in such warfare in recent times. Chisolm et al. [8] developed an infectious disease type of model for the diffusion of two competing opinions of a polarizing view which integrates outside elements and person-to-person connections. The model is derived from both epidemiological and competing species models to understand how members of America's Republican Party supported or were skeptical about the greenhorn candidates in the 2016 primary polls. The study developed and analyzed two skeptic, unexposed, proponent (SUP) models, one basic and one modified. The modified model is applied to a case study using the poll results for candidates Carson, Fiorina, and Trump to fit parameters.

Liu et al. [43] investigated the characteristics of a rumor propagation model with the concurrent spread of truth. They considered four population categories, namely the ignorants, the rumor spreaders, the truth spreaders and the stiflers. They assumed that the different spreaders have some likeliness of becoming stiflers when they interact or when they lose interest in spreading the information they know. The work by Mikhailov et al. [46] takes a deep dive into various dimensions of an information battle process in which two opposite views about a particular information item spread in the environment. They showed that in an information battle, the winning side is the one with the higher number of spreaders in the long run. Earlier works on rumor models like [9, 14, 15, 44, 49] were quite helpful in these studies. Feria et al. [16] developed a spreader-spreader and an exposed-spreader model based on the popular SEIR model for epidemics in order to understand the dynamics of rumor and truth spreading together in a population. Going by the first model, it is crucial to have a high removal rate of rumor spreaders so that truth can prevail. The second model requires that those who are exposed to rumor are quickly exposed to the truth afterwards in order to drown out the rumor.

In our context, a rejoinder is a reply issued to correct an incomplete or misleading piece of information that is already in circulation within a population. According to a study by Akpabio [2] on the direction of rejoinders in two of Nigeria's famous newspapers, it was discovered that the noble attribute of balanced reporting was lacking among some journalists. As a result, there were more of adversarial than mild rejoinders to some news items published about some individuals, organizations and governments. We hereby propose a rejoinder model in order to understand the nuances of counteracting pieces of information spreading within a population in a typical Internet-based social media setting. We propose a mathematical model that includes the possibility of human psychological and sociological tendencies like skepticism or deliberate negligence towards the corrective information such that some people continue to spread falsehood even after getting the accurate information. These kinds of behaviors are supported by various authors (including [13, 40]). Earlier on, rumor models, e.g.

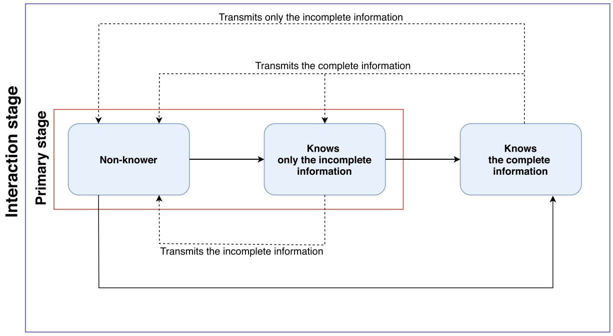

Fig. 1 Stage transition of individuals and their relation according to information transmission in the rejoinder model

the ones by [9, 34, 41, 44] were more suitable for person-to-person information transmission which follow the typical idea of networks.

2 Assumptions for modeling

Two pieces of information: The two pieces of information are such that the first one is incomplete and misleading whereas the second one is a rejoinder which is complete and corrective to the first.

Transmission and spread of information: The accurate piece of information is only shared alongside the wrong one such that a non-knower either gets to know and transmit only the first piece or both pieces of information at any given time. We assume that interactions on the Internet tend to happen very fast and do not significantly rely on the detailed structure of networks. So, complete mixing is assumed among those who are unexposed and transmitters who are either misinformed or correctly informed.

Two stages of the information spread: We consider the spread of information in two steps as shown in Fig. 1: first is the primary stage when the misinformation is introduced and begins to spread till a given time $t = t_s$; afterwards, we have the interaction stage after the complete piece of information is introduced at $t = t_s$. In the interaction stage, the complete piece of information spreads together with the wrong piece of information from time t_s which is the moment of rejoinder introduction.

Different attitudes towards the rejoinder: Of those who get to know the complete information after being misled initially, some get reinforced in transmitting the misleading information thereby going into the subpopulation V while the rest go directly into the subpopulation W_+ where they transmit the complete information (see Figs. 1 and 2). We assume that the wrongly informed people are expected to keep transmitting the incomplete and misleading information with probability b even after knowing the complete one (the transition to V). Further, we assume a strengthened motivation for V to be dogmatic and not want to stop transmitting the misleading information after getting to know the second piece. As such,

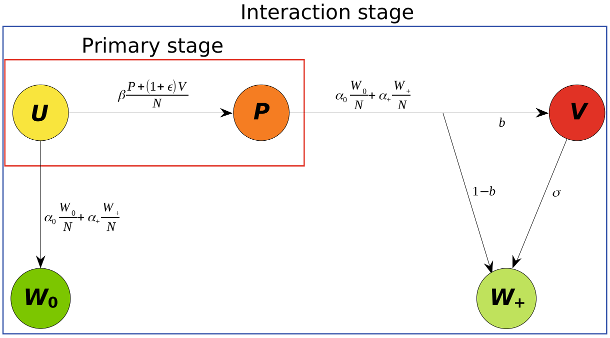

Fig. 2 State transition in the population dynamics for the spread of two interacting pieces of information. In the primary stage, we have the system (1) consisting of U and P, and at the interaction stage, we have (4)

the wrongly informed person comes to believe and transmit the complete information with a probability $1 - b$ after getting it (the transition to W_+).

3 Model

Primary stage

When the first piece of information, which is misleading, is the only one in circulation, we have the following dynamics of information spread for $t \in [0, t_s)$:

$$\frac{dU}{dt} = -\beta \frac{P}{N} U; \quad \frac{dP}{dt} = \beta \frac{P}{N} U \tag{1}$$

with initial condition $(U(0), P(0)) = (U_0, P_0)$. $U = U(t)$ is the population size of those who have not been exposed to the considered piece of information at time t, while $P = P(t)$ is the population size of those who come to know and transmit the misleading information at time t with transmission coefficient $\beta > 0$.

It is assumed that the total population size N is constant independently of time, so that the system (1) satisfies $U(t) + P(t) = N$ for any $t \in [0, t_s)$.

We can derive the following closed one-dimensional differential equation in terms of P from (1):

$$\frac{dP}{dt} = \beta \frac{P}{N}(N - P) \tag{2}$$

with initial condition $P_0 = N - U_0 > 0$. The solution is easily obtained as

$$P(t) = \frac{N}{1 + \frac{U_0}{P_0} e^{-\beta t}}, \tag{3}$$

so that $U(t) = N - P(t)$ for $t \in [0, t_s]$. This is monotonically increasing towards N in time.

Interaction stage

After the rejoinder is introduced at time $t = t_s$, we have the following dynamics of information spread for $t > t_s$ (see Fig. 2):

$$
\begin{aligned}
\frac{dU}{dt} &= -\beta\frac{P}{N}U - (1+\epsilon)\beta\frac{V}{N}U - \alpha_0\frac{W_0}{N}U - \alpha_+\frac{W_+}{N}U;\\
\frac{dP}{dt} &= \beta\frac{P}{N}U + (1+\epsilon)\beta\frac{V}{N}U - \alpha_0\frac{W_0}{N}P - \alpha_+\frac{W_+}{N}P;\\
\frac{dV}{dt} &= b\alpha_0\frac{W_0}{N}P + b\alpha_+\frac{W_+}{N}P - \sigma V;\\
\frac{dW_0}{dt} &= \alpha_0\frac{W_0}{N}U + \alpha_+\frac{W_+}{N}U;\\
\frac{dW_+}{dt} &= (1-b)\alpha_0\frac{W_0}{N}P + (1-b)\alpha_+\frac{W_+}{N}P + \sigma V,
\end{aligned}
\tag{4}
$$

with $U(t)+P(t)+V(t)+W_0(t)+W_+(t) = N$ for any $t > t_s$. Every parameter is a positive constant whose meaning is explained in the following part.

At this stage, as shown in Fig. 2, the non-knowers of U may get either the misleading information only or the complete one. After an individual of U gets the complete information, such a person is assumed to always come to transmit it; this is defined as the transition from the state U to W_0. $W_0(t)$ is the population size of those who have not been misled but transit from the state U to the state in which they know and transmit the complete (correct) information at time t.

$P(t)$ is the population size of those who know ONLY the misleading information and transmit it at time t. $W_+(t)$ is the population size of those who get misled before knowing and transmitting the complete information at time t. $V(t)$ is the population size of those who know the second piece of information but transmit ONLY the first piece of information at time t. It is now assumed that even such an individual, after getting the complete information, may get hardened in spreading the misleading information with a probability b, which is now defined as the transition from the state P to V.

The coefficient α_0 is for the transmission of complete information to U by those of W_0; α_+ is for the transmission of complete information to U by those of W_+; σ is the transition rate from V to W_+. It represents the change of thoughts to transmit the complete information after insisting on spreading the misleading first piece of information despite knowing the complete one before. ϵ is the increment of the transmission coefficient for the individuals of V, because of their tendency for information transmission psychologically stimulated or excited by receiving the second piece of information as mentioned in Section 2.

As the continuity between the primary and the interaction stages, we define

$$
U(t_s - 0) = \lim_{t\to t_s-0} U(t); \quad U(t_s + 0) = \lim_{t\to t_s+0} U(t) \tag{5}
$$

as well as the other variables. It marks the end of the primary stage (represented by $t_s - 0$) on the interval $[0, t_s)$ and the beginning of the interaction stage (given by $t_s + 0$). With θ $(0 < \theta < 1)$ representing the portion of people that get to learn about the complete information at time t_s, we define the initial condition at $t = t_s + 0$ as

$$
\begin{aligned}
&(U(t_s+0), P(t_s+0), V(t_s+0), W_0(t_s+0), W_+(t_s+0))\\
&\quad = ((1-\theta)U(t_s-0), (1-\theta)P(t_s-0), b\theta P(t_s-0), \theta U(t_s-0), (1-b)\theta P(t_s-0)),
\end{aligned}
\tag{6}
$$

where

$$\theta N = \theta[U(t_s - 0) + P(t_s - 0)] = V(t_s + 0) + W_0(t_s + 0) + W_+(t_s + 0). \tag{7}$$

The values of $P(t_s - 0)$ and $U(t_s - 0)$ are given by (3):

$$P(t_s - 0) = \frac{N}{1 + \frac{U_0}{P_0} e^{-\beta t_s}}; \quad U(t_s - 0) = N - P(t_s - 0). \tag{8}$$

With a set of non-dimensionally transformed variables and parameters $u := U/N$, $p := P/N$, $v := V/N$, $w_0 := W_0/N$, $w_+ := W_+/N$, $\tau := \beta t$, $\tau_s := \beta t_s$, $a_0 := \alpha_0/\beta$, $a_+ := \alpha_+/\beta$, and $c := \sigma/\beta$, the system (4) becomes the following non-dimensionalized system for $\tau > \tau_s$ (i.e. $t > t_s$):

$$\begin{aligned}
\frac{du}{d\tau} &= -pu - (1+\epsilon)vu - a_0 w_0 u - a_+ w_+ u;\\
\frac{dp}{d\tau} &= pu + (1+\epsilon)vu - a_0 w_0 p - a_+ w_+ p;\\
\frac{dv}{d\tau} &= b a_0 w_0 p + b a_+ w_+ p - cv;\\
\frac{dw_0}{d\tau} &= a_0 w_0 u + a_+ w_+ u;\\
\frac{dw_+}{d\tau} &= (1-b) a_0 w_0 p + (1-b) a_+ w_+ p + cv,
\end{aligned} \tag{9}$$

with $u(\tau) + p(\tau) + v(\tau) + w_0(\tau) + w_+(\tau) = 1$ for any $\tau > \tau_s$, and from (6) and (8), the initial condition

$$\begin{aligned}
&(u(\tau_s + 0), p(\tau_s + 0), v(\tau_s + 0), w_0(\tau_s + 0), w_+(\tau_s + 0))\\
&\quad = ((1-\theta)u(\tau_s - 0), (1-\theta)p(\tau_s - 0), b\theta p(\tau_s - 0), \theta u(\tau_s - 0), (1-b)\theta p(\tau_s - 0)),
\end{aligned} \tag{10}$$

where

$$p(\tau_s - 0) = \frac{1}{1 + \frac{U_0}{P_0} e^{-\tau_s}}; \quad u(\tau_s - 0) = 1 - p(\tau_s - 0). \tag{11}$$

4 Terminal state

The system (9) has three equilibrium states: $(u^*, p^*, v^*, w_0^*, w_+^*) = (1, 0, 0, 0, 0)$, $(0, 1, 0, 0, 0)$, and $(0, 0, 0, w_0^*, 1 - w_0^*)$. By the standard local stability analysis, it can be easily shown that the first two equilibrium states are always unstable. In Appendix 1, we show that the system necessarily converges to the third equilibrium state as indicated by the numerical calculations given in Fig. 3. Thus, the terminal state of the information spread in our model is characterized by the terminal population size w_0^* of non-misinformed people, or alternatively the size w_+^* of misinformed people.

The introduction of rejoinder is to correct the misinformation. However, it is far more important to ensure that as many people as possible escape from being misinformed in the first place. This makes prevention our ultimate goal so that the value of w_0^* at the equilibrium state is a critical estimator of the efficiency of rejoinder introduction. The smaller w_0^* is, the more unfavorable it is owing to the aim for rejoinder introduction.

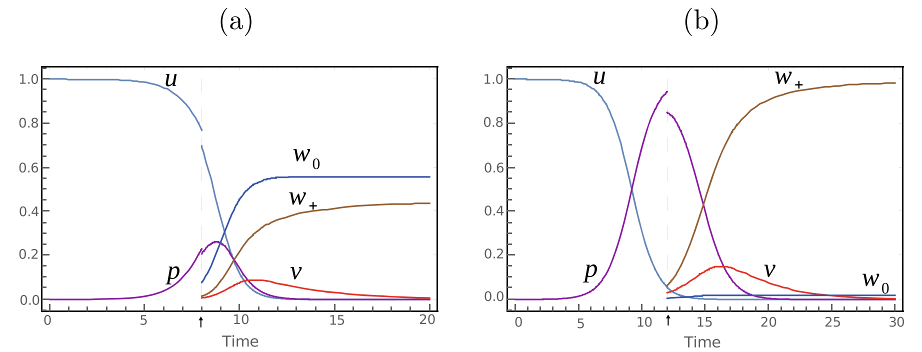

Fig. 3 Temporal variation showing the primary stage for the spread of the misleading information as represented by (11) and the interaction stage after the rejoinder is introduced at $\tau = \tau_s$ represented by the system (9) with (10). **a** $\tau_s = 8.0$; **b** $\tau_s = 12.0$. Commonly, $p(0) = 0.0001$, $u(0) = 1 - p(0)$, $\theta = 0.1$, $\epsilon = 0.05$, $a_0 = 2.00$, $a_+ = 1.10$, $b = 0.33$, $c = 0.30$

5 Instantaneous response to the rejoinder introduction

In this section, we consider the response of the misled population just after the rejoinder introduction at $\tau = \tau_s$. In many cases involving the news media, the efficiency of an operation taken against a wrong information is likely to be estimated/criticized by a relatively short-term response following the introduction of such an action. However, we will show later that such short-term response could not be an appropriate index to estimate the efficiency of rejoinder introduction in our model.

For $dp(\tau)/d\tau < 0$ at $\tau \to \tau_s + 0$ such that $p(\tau)$ declines immediately after the rejoinder is introduced, we can get the following necessary and sufficient condition from (9):

$$p(\tau_s + 0)[u(\tau_s + 0) - a_0 w_0(\tau_s + 0) - a_+ w_+(\tau_s + 0)] + (1 + \epsilon)v(\tau_s + 0)u(\tau_s + 0) < 0.$$

From the initial condition (10) with (11), this condition can be rewritten as follows:

$$\frac{1}{\theta} - \frac{1}{\theta_c} < a_+(1 - b)\frac{P_0}{U_0}\left(e^{\tau_s} - 1\right), \tag{12}$$

where

$$\theta_c := \frac{1}{a_+(1 - b)\frac{P_0}{U_0} + a_0 + (1 - b) - \epsilon b}. \tag{13}$$

If $0 < \theta_c \leq \theta$, the inequality (12) holds independently of τ_s, so that the introduction of the rejoinder is highly efficient to immediately reduce the size of the misinformed subpopulation. When $0 < \theta < \theta_c$, the misled population size increases or decreases depending on τ_s (see Fig. 4a-2). If $\theta_c < 0$, that is, if

$$a_+(1 - b)\frac{P_0}{U_0} + a_0 < \epsilon b - (1 - b),$$

such that ϵ or b is sufficiently large, there are sufficiently many misled people to actively spread the misleading information after knowing the complete one. In such a situation, the population size of misled individuals still increases after the introduction of the rejoinder independently of θ, if the rejoinder is introduced before the following critical moment τ_c, that is, when $\tau_s < \tau_c$ (Figs. 3a and 4a-1):

$$\tau_c := \ln \left| 1 + \frac{1}{a_+(1-b)} \frac{\theta_c - 1}{\theta_c} \frac{U_0}{P_0} \right|. \tag{14}$$

It is interesting that for sufficiently small τ_s in such a case, the misled population size increases, no matter how small. That is, no matter how early the rejoinder is introduced and no matter how large the portion of people who get the correct information at that moment, the misled population size continues to increase even after rejoinder introduction.

Figure 4b-1,2 shows the dependence of the instantaneous response on parameters ϵ and b, where $\epsilon_c := a_0 - 1/\theta$ and

$$b_c := 1 - \frac{\frac{1}{\theta} - a_0}{1 + a_+ \frac{P_0}{U_0} e^{\tau_s}}. \tag{15}$$

These results clearly imply that human psychological and sociological tendencies like skepticism or deliberate negligence towards the corrective information would contribute significantly to the instantaneous social response.

6 Efficiency of rejoinder introduction on the terminal state

The numerical calculation as shown in Fig. 5 indicates the existence of a specific range of τ_s for which a pronounced switch over of the value of w_0^* can be observed. Such a prominent range could not be identified for the dependence of the value of w_0^* on any other parameter, though the value of w_0^* depends on the other parameters in a much more moderate manner. As we can intuitively expect from the meanings of the parameters, the value of w_0^* monotonically increases in terms of θ, while it monotonically decreases in terms of b and ϵ (see numerical results given in Fig. 6).

This result implies that the aim of keeping people away from being misinformed is significantly achieved when the rejoinder is introduced earlier than a certain critical period. Otherwise, when the rejoinder is launched later, it has little or no impact in suppressing the population of misinformed people. The critical period for the moment of rejoinder introduction depends on the other parameters as numerically shown in Fig. 6a–c, though the dependence appears rather weak. This implies that the moment of rejoinder introduction itself is the most relevant factor which affects the extent to which misinformation spreads.

Our result conjectures that introducing the rejoinder after a critical period leads to little effect on the terminal population size of non-misinformed people. In contrast, introducing it earlier than the critical period may result in a rather large terminal population size of non-misinformed people. Releasing the rejoinder within the critical period, that is, the specific critical range of τ_s, the terminal population size of non-misinformed people is sensitively determined by the actual moment, so that the earlier introduction of rejoinder can result in significantly larger terminal population size of non-misinformed people.

7 Concluding remarks

The model proposed by Liu et al. [43] shows that when there is increased spread of truth-rumor, the propagation of false-rumor can be obliterated; it was also seen that the false-rumor and the truth-rumor can coexist for long given certain circumstances. To some extent, these

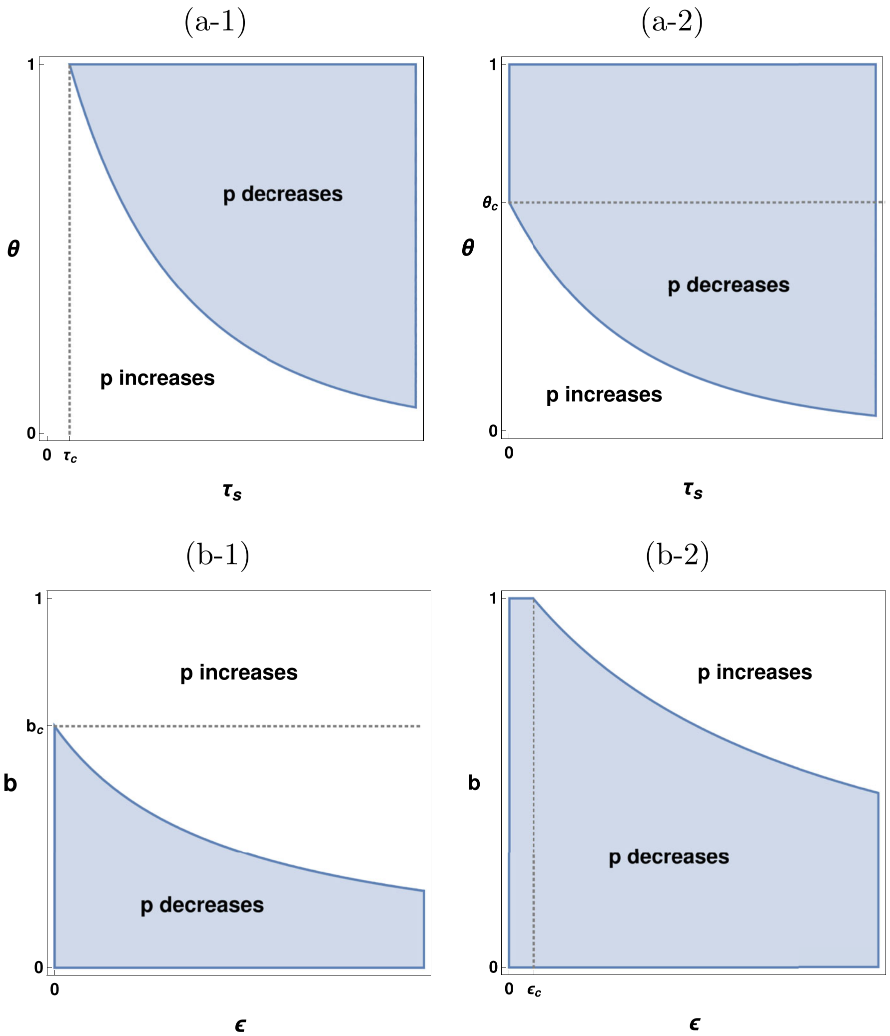

Fig. 4 Parameter dependence of the instantaneous response to the introduction of rejoinder, based on the inequality (12): **a-1** $\theta_c < 0$ or $\theta_c \geq 1$; **a-2** $0 < \theta_c < 1$; **b-1** $a_0\theta < 1$; **b-2** $a_0\theta \geq 1$. The shaded parts are regions of decrease while the unshaded parts are regions of instantaneous increase of the misled population size just after the introduction of rejoinder

findings tend to agree with the results from our model since the introduction of a rejoinder can accelerate the elimination of misleading information. Feria et al. [16] established the importance of the early introduction of truth by relevant spreaders to make it endemic in a population. This seems to correspond with the early rejoinder introduction in our model, with similar effect. The instantaneous response of the population dynamics just after the rejoinder introduction does not match up to the consequence of interaction between pieces of information as shown by Figs. 4 and 6. Although the earlier introduction of rejoinder can result in the more preferable consequence of saving people from being misinformed in the long run, it tends to cause such an instantaneous response that the misled people still increase after the

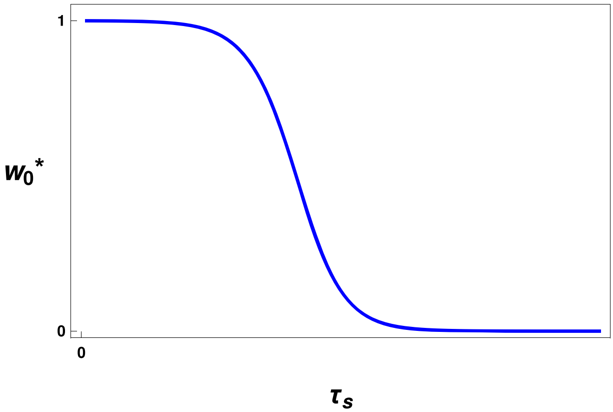

Fig. 5 Dependence of the equilibrium value w_0^* on the moment of rejoinder introduction τ_s. A numerical result with $\theta = 0.10$, $b = 0.33$, $\epsilon = 0.05$, $p(0) = 0.0001$, $u(0) = 1 - p(0)$, $a_0 = 2.00$, $a_+ = 1.10$, and $c = 0.30$. w_+^* is given by $1 - w_0^*$

rejoinder is introduced. This implies that the short-term response to rejoinder introduction would not be an appropriate index about its efficiency eventually.

The rejoinder model has some similarities with the classical Kermack-McKendrick SIR model about the population dynamics of transmissible diseases. For instance, the population sizes of non-knowers (U), knowers and transmitters of the misleading piece of information (P, V), knowers and transmitters of the complete information (W_0, W_+) in the rejoinder model correspond respectively with the population sizes of susceptibles (S), infectives (I), removed (R) in the SIR model. The population size W_0 can be considered, for example, as those who are shielded from infection through vaccination while W_+ are like those who develop natural immunity having been previously infected. Though the two models are similar in structure, they are different in dynamics. This is demonstrated by the fact that the population size R in the SIR model has no effect on the population sizes S and I with respect to the epidemic dynamics. However, in the rejoinder model, the population sizes W_0, W_+ have direct impact on the population sizes U, P and V.

Our model shows the effect of mass action that better estimates interactions on the Internet in comparison with earlier rumor models which are more biased towards network theory. This is because people are not necessarily connected following the traditional theory of networks [7]. On social media platforms, there are lots of misleading information about governmental and non-governmental organizations. Sometimes, there are also misleading information from such institutions in form of propaganda [36]. So, it has become imperative to be able to tell apart correct and wrong information. It has been widely agreed that the problems of misinformation and disinformation can be mitigated by promoting information literacy through multidisciplinary collaborations (see [37, 38, 63]).

Since the acceptance or neglect of a piece of information can be regarded as dependent on the decision-making of each person in the population, the individual heterogeneity could significantly determine the nature of the dynamics of information spread. Granovetter [24, 25] presented a well-known novel idea, called *Granovetter's threshold model*, to theoretically consider the process of spreading information. This has been applied to a variety of areas like diffusion of innovation, public protests, migration, voting, market trends, international relations, and information spread [4, 11, 27, 31, 33, 39, 59].

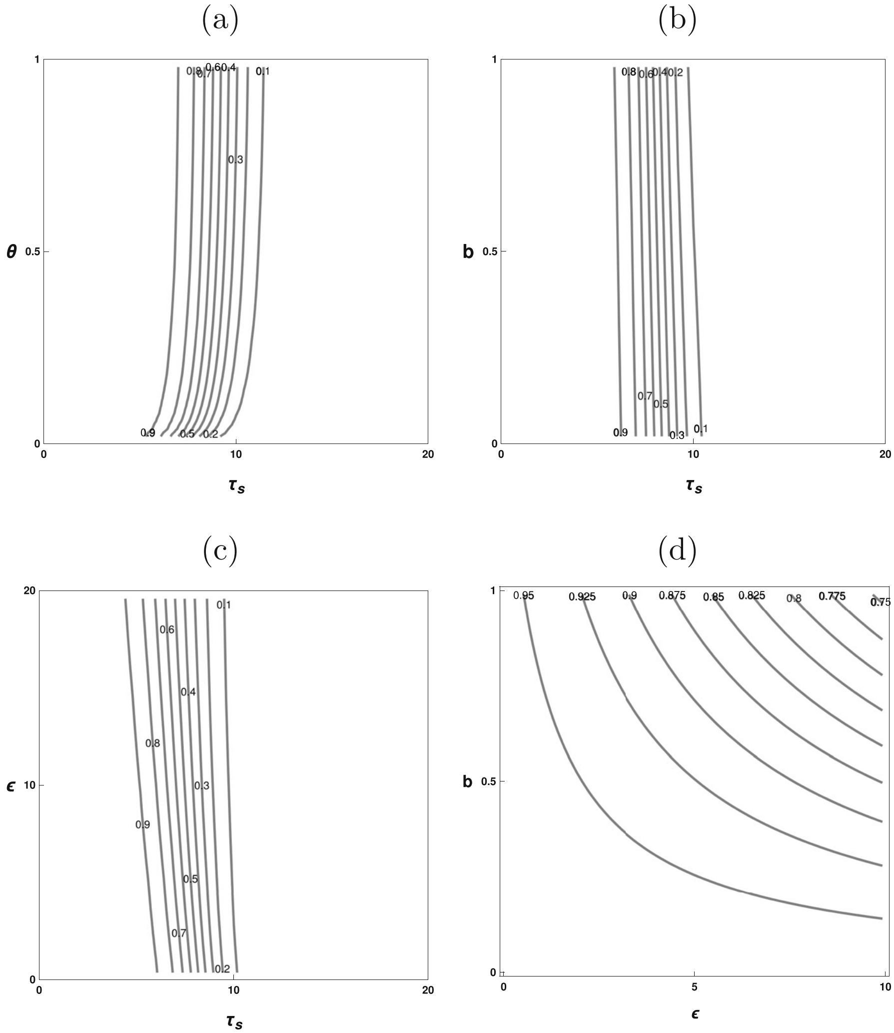

Fig. 6 Contour plot of the dependence of w_0^* on **a** (τ_s, θ) with $b = 0.33$ and $\epsilon = 0.05$; **b** (τ_s, b) with $\theta = 0.10$ and $\epsilon = 0.05$; **c** (τ_s, ϵ) with $\theta = 0.10$ and $b = 0.33$; **d** (ϵ, b) with $\theta = 0.10$ and $\tau_s = 5$. Numerically drawn commonly with $p(0) = 0.0001$, $u(0) = 1 - p(0)$, $a_0 = 2.00$, $a_+ = 1.10$, and $c = 0.30$

He regards information spread as a collective behavior under certain circumstances where people have to make one of two distinct choices such that the merit or demerit in each choice depends on the number of individuals who decide for or against it. When the number of individuals who have taken the decision reaches a threshold, the advantages of taking the decision begin to outweigh the disadvantages for a given individual. Granovetter and Soong [26] showed the importance of threshold modeling as lying in the not-so-simple connection between individual choices and overall steady results. The Granovetter's threshold model was, originally, not a mathematical model but a conceptual model.

Various mathematical and computational approaches abound in literature for studying collective behaviors with an extension of Granovetter's idea. Castellano et al. [6] highlighted the relevance of statistical physics to other areas of learning apart from physics. Assuming a network that is random and non-finite with weak connections, Whitney [64] tried to understand diffusion (of information or innovation) on a network using generating functions. The theory proposed is based on a threshold rule which ensures that a node only changes state after a fraction of nearby nodes, surpassing a particular limit, have previously flipped over. Akhmetzhanov et al. [1] extensively applied the Granovetter's idea for a network of individuals in a square lattice with each one having a state and a specified threshold for change in behavior. A utility-psychological threshold model based on the Granovetter's threshold model was introduced by [42]. The critical shift in phase of group behavior is studied by taking into account rational utility and psychological thresholds under the influence of space and intensity of social network [60, 61, 65]. We find other interesting approaches and methods in [18–20, 45, 48, 51, 52, 56, 57, 66]. Although most of the previous models described the conceptual process in Granovetter's threshold model, and they may not be regarded as reasonable population dynamics models for temporal variation of the number or frequency of acceptors of a matter spreading in a population, Dansu and Seno [10] revisited Granovetter's idea of collective behavior, and refined the mathematical modeling to derive a mathematically accurate and reasonable population dynamics model with an ordinary differential equation which gives the temporal variation of the number or frequency of individuals who have accepted an information spreading in a population. The analytical results on their model showed that the distribution of heterogeneous individuality in a population determines the success or failure of the spread of a piece of information in a population, as conceptually discussed by Granovetter [24, 25].

As implied by those previous works, it is necessary to have a reasonable population dynamics model to discuss the nature of temporal change in social situations with temporal variations in their subpopulation sizes regarding a spreading matter in a population. In the future, we hope to extend our model to accommodate human heterogeneity in the handling of information.

Funding EJD was supported by the Japanese Government (Monbukagakusho: MEXT) Scholarship Number 170805. HS was supported in part by the JSPS KAKENHI Grant Number 18K03407.

Appendix

Convergence to the equilibrium state

Since $u(\tau_s+0) > 0$ and $w_0(\tau_s+0) > 0$ from the initial condition assumed in our model, and since $dw_0/d\tau > 0$ for any $\tau \in (\tau_s, \infty)$, we have $w_0(\tau) > 0$ for any $\tau > \tau_s$. This argument can also be applied for w_+, so that $w_+(\tau) > 0$ for any $\tau > \tau_s$. As long as u is positive for (9) and any of p, w_0 and w_+ is also positive, $du/d\tau$ is always negative. This is contrary if $u \to u^* > 0$, as such, $u \to 0$. So, as $u \to 0$, the right hand side of $dp/d\tau$ must become negative since the first term becomes significantly small compared to the other terms, for sufficiently large $\tau > \tau_s$. It is now clear that p decreases such that $p \to 0$ and $v \to 0$. From this analysis, we have $(u, p, v, w_0, w_+) \to (0, 0, 0, w_0^*, w_+^*)$ as the convergence state.

Since $u + p + v + w_0 + w_+ = 1$ for any $\tau > \tau_s$, the convergence as $\tau \to \infty$ means that w_0 and w_+ converge to some positive values w_0^* and w_+^* such that $w_0^* + w_+^* = 1$. The convergent value w_0^* or w_+^* depends on the initial condition at $\tau = \tau_s + 0$, which we could not determine analytically.

As an extremal mathematical supposition, if $u(\tau_s - 0) = 1$ and $p(\tau_s - 0) = 0$, then $u(\tau_s + 0) \leq 1$ and $p(\tau_s + 0) = 0$ so $dp/d\tau = 0$ and $p(\tau) = 0$ for any $\tau > \tau_s$ at the interaction stage. This means that $dw_+/d\tau = 0$ for any $\tau > \tau_s$ at this stage since $w_+(\tau) = w_+(\tau_s + 0) = 0$ for any $\tau > \tau_s$, so $w_+^* = 0$ and $w_0^* = 1$. On the other hand, if we assume the extremum situation where $u(\tau_s - 0) = 0$ and $p(\tau_s - 0) = 1$, then $u(\tau_s + 0) = 0$ and $p(\tau_s + 0) \leq 1$ since $du/d\tau = 0$ and $u(\tau) = 0$ for any $\tau > \tau_s$ at the interaction stage. This means that $dw_0/d\tau = 0$ for any $\tau > \tau_s$ at this stage since $w_0(\tau) = w_0(\tau_s + 0) = 0$ for any $\tau > \tau_s$ so $w_0^* = 0$. Overall, $w_0^* \in (0, 1]$.

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