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Some Mathematical Considerations on Two-mode Searching II

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Along the lines of a previous paper (Seno [2]), making use of an intuitive model, we consider mathematically the relation between the efficiency of searcher's two-mode searching behavior and the target's patchy distribution, and discuss the strategic adaptability of two-mode searching. In this paper, the model is constructed by a time-discrete stochastic process on $\mathbf{S}^1$, that is, on a circle. It can be regarded as a modification of Model 2 analyzed in Seno [2]. Differently from Model 2, the searcher's present location is assumed to be influenced by the past passage configuration. This modification yields some particular results for the present model.

Also in the present model, if the patch size becomes sufficiently small, a two-mode searching behavior is strategically adaptable for the searcher. In this model, two-mode searching behavior has high strategic adaptability. Moreover, two-mode searching with an outstanding behavior change is strategically rather adaptable. As for the target's distribution, it appears that a particular patchy distribution is likely to be adopted, depending on the searcher's searching strategy. This result obviously indicates that the target's distribution may be adopted as its evolutionary strategy against the searcher, like a relation between a patchy distributed prey and its predator.

Key words: searching, mathematical model, Markov chain

1. Introduction

In the previous paper (Seno [2]), making use of two simple mathematical models, we considered a coevolutionary game between the searcher's searching behavior and the target's patchy distribution, and demonstrated a strategic adaptability of two-mode searching (i.e., area-concentrated search) depending on the target's distribution strategy. In this paper, we shall consider a coevolutionary game again with an intuitive model constructed by a time-discrete stochastic process on $\mathbf{S}^1$, that is, on a circle. The model can be regarded as a modification of Model 2 of the quoted paper. The searcher's present location is assumed to be influenced by the past passage configuration, which is an essentially different assumption from that for Model 2 in the previous paper. We shall see that this assumption carries such a mathematical complexity that the model is not easily analytically tractable anymore. We shall apply the Monte Carlo method to obtain some numerical results, and derive some significative figures to illustrate our argument on the coevolutionary game. The results for the present model show some particular features different from those of Model 2.

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2. Model and Analysis

The model is considered on $\mathbf{S}^1$, that is, on a circle (Fig. 1). The modelling space $\mathbf{S}^1$ can be regarded as a mathematical translation of the space $\mathbf{R}^1$ where patches are all identical and regularly distributed. The searcher is assumed not to be able to distinguish a visited patch from an unvisited one. In each patch, the targets are assumed to be regularly distributed. Moreover, as the found target is not removed, it is assumed that the searcher cannot distinguish the found target from the encountered one.

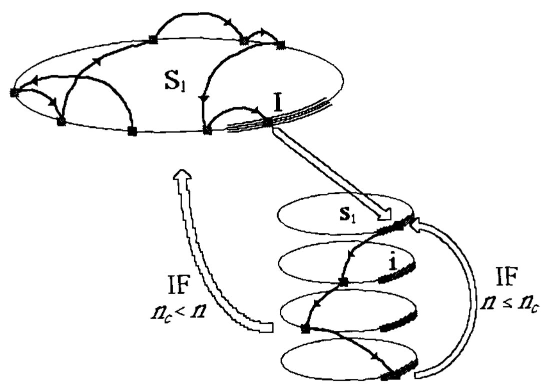

Fig. 1. Scheme of model. The patch-searching process is terminated when the searcher enters the region I on S_1. The target-catching process is subjected to a fixed-giving up step number strategy with n_c. For a more detailed explanation, see the text.

For a fixed number of targets within a patch, the higher density of targets will be assumed to imply the smaller patch size. The efficiency of searching a patch and that of catching targets in a patch are not independent.

Our model consists of two processes on $\mathbf{S}^1$: i) patch-searching process; ii) target-catching process. In each of these processes, the searcher is viewed as a point moving discretely with a stochastic process. Following a discrete time scale, the searcher discretely changes its site on $\mathbf{S}^1$ by each step. The distance between a site and the following site is assumed to be an exponential random variable. The direction of each step is selected at random, that is, with the probability $1/2$ the searcher jumps to the next site in the clockwise or in the anticlockwise direction. The switching rule between two processes is as follows (Fig. 1): The patch-searching process is terminated when the searcher enters a region on $\mathbf{S}^1$. This is the moment when the searcher catches the first target. The target-catching process continues until the number of steps between a catch and the next overcomes a critical number n_c. This means that the searcher takes a "fixed-giving up time (i.e., number of steps)" strategy (for example, see Iwasa et al. [1]).

The searching efficiency E is defined as follows:

$$E \equiv \frac{M}{n_1 + n_2},$$

where n_1 denotes the number of steps taken in the patch-searching process, n_2 that taken in the target-catching process for catching M targets. We shall investigate the optimal strategy to realize the highest efficiency for a fixed patch's quality (distances between nearest-neighbor patches and between nearest-neighbor targets).

As for the strategic tendency of targets' distribution, we shall consider two contrast types: "the counter-behaving target" and "the cooperative-behaving target". The distribution is respectively directed to make the mean efficiency as low as possible for the former and as high as possible for the latter.

Mathematical Statement

Patch-Searching Process: We consider this process on a circle of length A. On this space, there is a connected region (an arc) of length l, which represents the zone of patch. This corresponds to the situation in which the patch (segment) of length l is regularly distributed on $(-\infty, +\infty)$ of $\mathbf{R}^1$ with distance $A - l$ between nearest-neighbor patches. At first, we must select the initial site x_0 of the searcher out of the patch. It is assumed that the initial site is uniformly distributed out of the patch. The next searcher's step in subjected to the exponential distribution with expected value λ_1, that is, with probability density function given by:

$$f_1(\Delta x) = \frac{1}{\lambda_1} \exp\left(-\frac{\Delta x}{\lambda_1}\right) \tag{1}$$

For the patch-searching process, we shall use the following notations:

$x \in \mathbf{S}_1 \equiv [0, A] \bmod A$

$\mathbf{I}$:	zone of patch, $\mathbf{S}_1 \supset \mathbf{I} \equiv (A - l, A) \bmod A$
$\mathbf{G}$:	zone out of patch, $\mathbf{S}_1 \supset \mathbf{G} = \mathbf{S}_1 - \mathbf{I} \equiv [0, A - l] \bmod A$
$P_1{}^{\mathrm{in}}(x)$:	probability of the searcher's entrance by one step into the patch from a point x out of the patch, independently of the point reached in the patch
$P_1{}^g(x \to y)$:	probability of the searcher's one step from x to y out of the patch
$P^n{}_{\mathbf{x}}(x_0)$:	probability of the searcher's entrance by n steps into the patch from the initial point x_0 out of the patch with a configuration $\mathbf{x} \equiv (x_0, x_1, \ldots, x_{n-1})$ in $\mathbf{G}$, independently of the point reached in the patch
$\langle n_1 \rangle$:	expected number of steps for the searcher to enter firstly the patch, averaged with respect to the initial point and the configuration of searching.

With these notations, the following two mathematical relations follow from the Markov nature of the process:

$$P^n{}_{\mathbf{x}}(x_0) = \prod_{k=1}^{n-1} P_1^g(x_{k-1} \to x_k) P_1{}^{\text{in}}(x_{n-1}) \tag{2}$$

and

$$\langle n_1 \rangle = \sum_{n=1}^{+\infty} \int_{\mathrm{G}} dx_0 \int_{\mathrm{G}} dx_1 \int_{\mathrm{G}} dx_2 \ldots \int_{\mathrm{G}} dx_{n-1} n \cdot P^n{}_{\mathbf{x}}(x_0). \tag{3}$$

By a cumbersome calculation based on the use of the probability density function (1), we find the followings (Appendix):

$$P_1^{\text{in}}(x) = \sinh\left(\frac{l}{2\lambda_1}\right) \sinh\left(\frac{2x - A + l}{2\lambda_1}\right) \operatorname{sech}\left(\frac{A}{2\lambda_1}\right), \tag{4}$$

$$P_1^g(x \to y) = \frac{1}{2\lambda_1} \cosh\left(\frac{A - 2|x - y|}{2\lambda_1}\right) \operatorname{sech}\left(\frac{A}{2\lambda_1}\right). \tag{5}$$

Then, through the relation (2), we obtain:

$$\begin{aligned} P^n{}_{\mathbf{x}}(x_0) = & \left(\frac{1}{2\lambda_1}\right)^{n-1} \left\{\operatorname{sech}\left(\frac{A}{2\lambda_1}\right)\right\}^n \sinh\left(\frac{l}{2\lambda_1}\right) \times \\ & \cosh\left(\frac{2x_{n-1} - A + l}{2\lambda_1}\right) \prod_{k=0}^{n-2} \cosh\left(\frac{A - 2|x_k - x_{k+1}|}{2\lambda_1}\right). \end{aligned} \tag{6}$$

Target-Catching Process: The searcher is assumed to catch the first target at the moment when it begins the target-catching process. We shall assume that the total number of targets is N in each patch, and that the targets are regularly distributed in it. Therefore, the following relation is assumed to hold:

$$Nd = l, \tag{7}$$

where d is the distance between the nearest-neighbor targets. In addition, we shall assume an effective distance r between the searcher and the target, within which the searcher can find and catch the target. Thus, only if the searcher arrives at a site whose distance from the target is less than r, it catches the target. For mathematical simplicity, we shall ignore any handling time: In other words, the searcher is assumed to catch the target at the moment when it reaches a site whose distance from the target is less than r.

Following the above modelling assumptions, the searcher's initial site in the target-catching process is the center of target's region which has length $2r$ (see Fig. 1). Now we shall regard the center point of target's region as the origin on

$\mathbf{S}^1$. Further, after catching a target, the searcher is assumed to begin always its next target-catching process from the center of target's region. The searcher's step is subjected to the exponential distribution with expected value λ_2, that is, with probability density function:

$$f_2(\Delta x) = \frac{1}{\lambda_2}\exp\left(-\frac{\Delta x}{\lambda_2}\right). \tag{8}$$

As mentioned before, the searcher is assumed to take a fixed-giving up step strategy: If the searcher fails to find any target after n_c steps, the searcher gives up its target-catching process then changes its behavior to the patch-searching process. In nature, the searcher may stochastically go out of the patch, and the smaller the patch size is, the larger such probability must be. This aspect shall be neglected in our model for mathematical simplicity, though it is likely that it may be more or less critical for our following argument.

Below we list up the notations for the target-catching process:

$z \in \mathbf{s}_1 \equiv [0, d]$ mod d

$\mathbf{i}$:	target's region, $\mathbf{s}_1 \supset \mathbf{i} = \mathbf{i}_1 \cup \mathbf{i}_2 \equiv [0, r) \cup (d - r, a)$ mod d
$\mathbf{g}$:	region out of target, $\mathbf{s}_1 \supset \mathbf{g} = \mathbf{s}_1 - \mathbf{i} \equiv [r, d - r]$ mod d
n_c:	fixed-giving up step, i.e., the behavior-switching step number in the target-catching process
$P_2{}^{\mathrm{in}}(z)$:	probability of the searcher's catching the target by one step from a point z out of the target, independently of the point reached in the target's region
$P_2{}^{\mathrm{out}}(z)$:	probability of the searcher's one step to a point z from the origin that is the center of target's region
$P_2{}^{g}(z \to w)$:	probability of the searcher's one step from z to w out of the target's region
$P^n{}_{\mathbf{z}}$:	probability of the searcher's catching the target by n steps with a configuration $\mathbf{z} \equiv (z_0, z_1, \ldots, z_{n-1})$ in $\mathbf{g}$
$P^n{}_{\langle \mathbf{z} \rangle}$:	probability of the searcher's catching a target after n steps, averaged with respect to the configuration of searching
$P^c{}_{\langle \mathbf{z} \rangle}$:	probability of the searcher's catching a target by less than n_c steps, averaged with respect to the configuration of searching
$q^c{}_{\langle \mathbf{z} \rangle}$:	probability of the searcher's failure to catch any target after n_c steps, averaged with respect to the configuration of searching and the point reached after n_c steps
$\langle n_2 \le n_c \rangle$:	expected number of steps for the searcher to catch another target after catching one, averaged with respect to the configuration of searching, conditional on the number of steps being equal to or less than the fixed-giving up step number n_c
$\langle n_2 \rangle$:	expected total number of steps in the target-catching process before the searcher gives it up
$\langle M \rangle$:	expected number of targets caught in the target-catching process before the searcher gives it up.

With these notations, the following relations are found (as for some non-trivial relations, see Appendix):

$$P^n{}_{\mathbf{z}} = P_2{}^{\mathrm{out}}(z_1) \prod_{k=2}^{n-1} P_2^g(z_{k-1} \to z_k) P_2{}^{\mathrm{in}}(z_{n-1}) \tag{9}$$

$$P^n{}_{\langle \mathbf{z} \rangle} = \int_{\mathbf{g}} dz_1 \int_{\mathbf{g}} dz_2 \dots \int_{\mathbf{g}} dz_{n-1} P^n{}_{\mathbf{z}} \tag{10}$$

$$P^c{}_{\langle \mathbf{z} \rangle} = \sum_{n=1}^{n_c} P^n{}_{\langle \mathbf{z} \rangle} \tag{11}$$

$$q^c{}_{\langle \mathbf{z} \rangle} = 1 - P^c{}_{\langle \mathbf{z} \rangle} \tag{12}$$

$$\langle n_2 \le n_c \rangle = \sum_{n=1}^{n_c} \frac{n P^n{}_{\langle \mathbf{z} \rangle}}{P^c{}_{\langle \mathbf{z} \rangle}} \tag{13}$$

$$\langle M \rangle = \frac{1}{1 - P^c{}_{\langle \mathbf{z} \rangle}} \tag{14}$$

$$\langle n_2 \rangle = \{\langle M \rangle - 1\} \langle n_2 \le n_c \rangle + n_c. \tag{15}$$

Making use of (8), we obtain (cf. Appendix):

$$P_2^{\mathrm{in}}(z) = \mathrm{sech}\left(\frac{d}{2\lambda_2}\right) \sinh\left(\frac{r}{\lambda_2}\right) \cosh\left(\frac{2z - d}{2\lambda_2}\right) \tag{16}$$

$$P_2^{\mathrm{out}}(z) = \frac{1}{\lambda_2} \mathrm{sech}\left(\frac{d}{2\lambda_2}\right) \cosh\left(\frac{2z - d}{2\lambda_2}\right). \tag{17}$$

In analogy with the result worked out for the patch-searching process, we also obtain:

$$P_2{}^g(z \to w) = \frac{1}{2\lambda_2} \cosh\left(\frac{d - 2|w - z|}{2\lambda_2}\right) \mathrm{sech}\left(\frac{d}{2\lambda_2}\right). \tag{18}$$

Finally, we are led to the following expression:

$$\begin{aligned} P^n{}_{\mathbf{z}} = 2 \left(\frac{1}{2\lambda_2}\right)^n & \left\{\mathrm{sech}\left(\frac{d}{2\lambda_2}\right)\right\}^n \sinh\left(\frac{r}{\lambda_2}\right) \cosh\left(\frac{d - 2z_1}{2\lambda_2}\right) \times \\ & \cosh\left(\frac{d - 2z_{n-1}}{2\lambda_2}\right) \prod_{k=1}^{n-2} \cosh\left(\frac{d - 2|z_{k+1} - z_k|}{2\lambda_2}\right). \end{aligned} \tag{19}$$

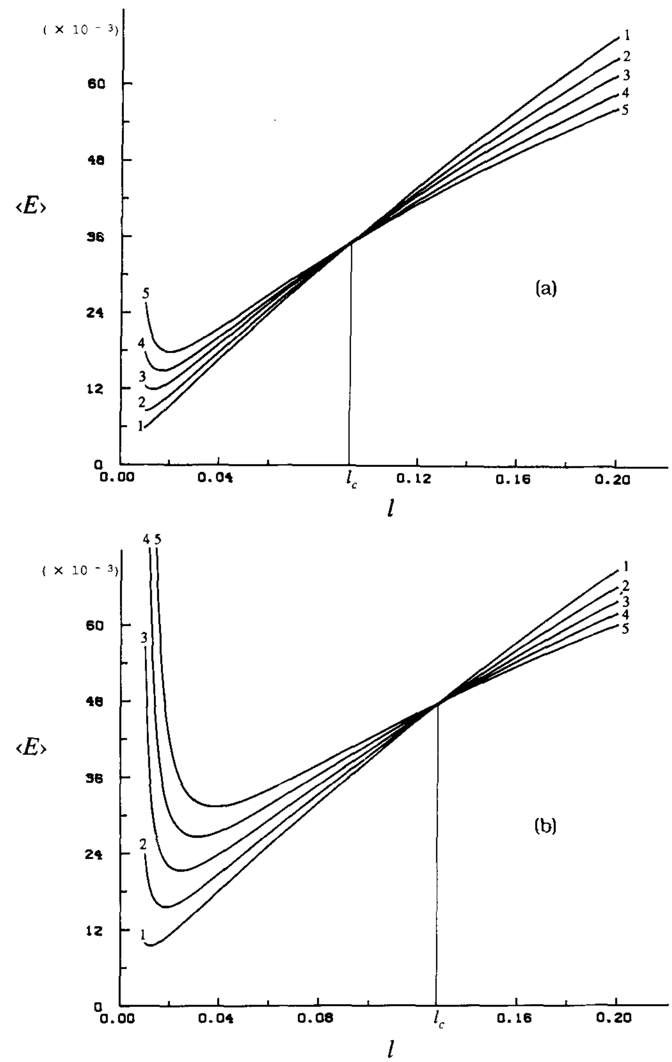

Fig. 2. l-$\langle E\rangle$ relation. $\lambda_2 = 0.1$. a) $N = 8$; b) $N = 15$. Attached integer on each curve indicates the giving up number n_c.

Efficiency: With the above results, we shall discuss the mean efficiency given by:

$$\langle E\rangle = \frac{\langle M\rangle}{\langle n_1\rangle + \langle n_2\rangle}. \tag{20}$$

We were unable to investigate this mean efficiency with any purely analytical way, because of the difficulty to integrate (3) and (10). Thus, we have investigated it by numerical calculations including numerical integrations. Practically we have used the Monte Carlo method to calculate multidimensional integrals. On the grounds of our numerical results, in the next section we shall discuss a strategic relation between the searcher's searching behavior and the target's distribution.

Analysis

With the mathematical formulation obtained in the previous section, we can find some characteristics of the efficiency (20) in some figures drawn by tiresome numerical calculations. We shall select some parameters and give them fixed values: $\lambda_1 = 1.0, A-l = 2.5, r = 0.0002$. We shall focus on the patch size l, the total number N of targets within a patch, the mean step length λ_2 of searcher, and the giving up step number n_c. l and N belong to the strategy of target's distribution, and λ_2 to that of searcher's behavior. Note that, since our attention is on an "area-concentrated search", we shall consider only the case when $\lambda_2 < \lambda_1$.

Figure 2 shows the relation among l, $\langle E\rangle$, and n_c. Note that each curve has a unique minimum efficiency point at $l = l^*$. $\langle E\rangle$ reaches 1 for a sufficiently small l less than l^*. For such an l, $\langle n_1\rangle$ becomes very large because the patch size is so small that it is very difficult for the searcher to find the patch. On the other hand, for a fixed number of targets within a patch, a small patch size leads to a high density of targets within the patch. This follows from the relation (7). Therefore, for a sufficiently small l less than l^*, $\langle M\rangle$ and $\langle n_2\rangle$ become very large because the searcher can easily find the target in the patch due to its high density and should continue the target-catching process for a very long period. Indeed, when the patch size l reaches $2rN$, the distance between the nearest-neighbor targets reaches $2r$. Then, in the target-catching process, any searcher's step tends to be in the target's region. Therefore, as $l \downarrow 2rN$, $\langle M\rangle$ and $\langle n_2\rangle$ become *infinitely* large, while $\langle n_1\rangle$ is *finitely* large because the region g has still a positive measure on $\mathbf{s}_1$. Further, $P^c{}_{\langle \mathbf{z}\rangle}$ and $\langle n_2 \le n_c\rangle$ respectively reach 1, in agreement with their definitions. Therefore, the efficiency reaches 1 when l reaches $2rN$. This is unfavorable for the counter-behaving target and favorable for the cooperative-behaving one. Thus, the counter-behaving target must take a patch size l larger than $2rN$, while the cooperative-behaving target may take a patch size $2rN$ with which every nearest-neighbor targets touch each other in the patch.

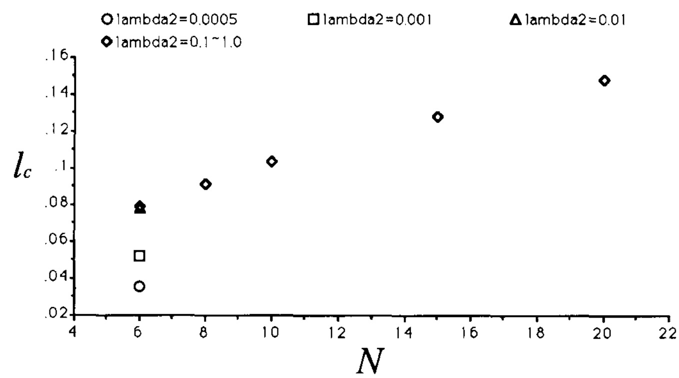

Fig. 3. N-l_c relation.

CONJECTURE 1. *Keeping other parameters fixed, there is a unique l, say l^*, at which the efficiency $\langle E\rangle$ takes its minimum value, say $\langle E\rangle_{\min}$. $\langle E\rangle_{\min}$ is mono-*

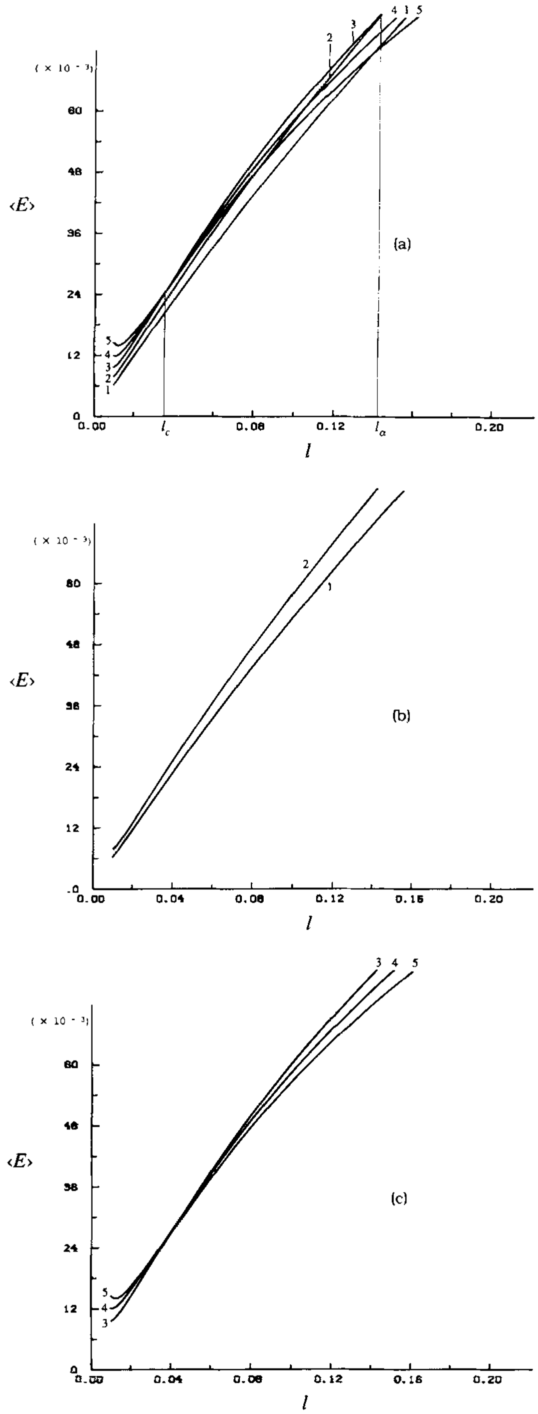

Fig. 4. l-$\langle E\rangle$ relation for some nc. $\lambda_2 = 0.0005$. a) for $n_c = 1, 2, 3, 4$, and 5: b) only for $n_c = 1$ and 2; c) only for $n_c = 3, 4$, and 5. l_c and l_α are marked in (a).

tonically decreasing with respect to n_c, *monotonically increasing with respect to* N. $\langle E \rangle$ *is monotonic with respect to* l *as follows:*

$$\frac{\partial \langle E \rangle}{\partial l} < 0 \qquad \text{when } l < l^*$$

$$\frac{\partial \langle E \rangle}{\partial l} > 0 \qquad \text{when } l^* < l.$$

In case of Fig. 2, all the efficiency curves have a unique common cross point at $l = l_c$, independently of n_c. Note that the efficiency curve is monotonically increasing with respect to n_c for any $l < l_c$ and decreasing for any $l_c < l$. This value l_c is a monotonically increasing function of n (Fig. 3). However these characteristics are vulnerable for a sufficiently small λ_2 and a sufficiently small n_c as for example shown in Fig. 4. We include this characteristic by the following conjecture:

CONJECTURE 2. *For a fixed* n_c, *say* $n_{c,\text{fixed}}$, *there is such a critical value of* λ_2, *say* λ_2^c *that, if* $\lambda_2^c < \lambda_2$, *there is a unique* l, *say* l_c, *such that the efficiency is not contributed by* n_c $(\geq n_{c,\text{fixed}})$: $\partial \langle E \rangle / \partial n_c |_{l=l_c} = 0$. λ_2^c *is a monotonically decreasing function of* $n_{c,\text{fixed}}$. *Thus, for* $\lambda_2 > \lambda_2^c|_{n_c=1}$, *a unique* l_c *exists for all positive integer* n_c.

Now and hereafter n_c is expandedly considered as a real number. Focusing on the case when $\lambda_2^c < \lambda_2$, we are led to some more mathematical conjectures.

CONJECTURE 3. *With* $\lambda_2^c < \lambda_2$ *and such* n_c *that there exists* l_c, *the following tendency is satisfied:*

$$\frac{\partial \langle E \rangle}{\partial n_c} > 0 \qquad \text{when } l < l_c$$

$$\frac{\partial \langle E \rangle}{\partial n_c} = 0 \qquad \text{when } l = l_c$$

$$\frac{\partial \langle E \rangle}{\partial n_c} < 0 \qquad \text{when } l_c < l.$$

Moreover, from Fig. 2 and Fig. 3, this l_c has the following characteristics:

CONJECTURE 4. *One has always* $l^* < l_c$. *Moreover, for* $\lambda_2^c < \lambda_2$, λ_2 *contributes little to* l_c, *that is,* $\partial l_c / \partial \lambda_2 \sim 0$. l_c *is a monotonically increasing function of* N.

As for l^*, from Fig. 5 and Fig. 6, we find the following:

CONJECTURE 5. l^* *is affected little by* λ_2 *such as* $\lambda_2^c < \lambda_2$, *that is,* $\partial l^* / \partial \lambda_2 \sim 0$, *but monotonically increasing with respect to* n_c, *that is,* $\partial l^* / \partial n_c > 0$.

Now let us turn our attention on the complex case when $\lambda_2 < \lambda_2^c$. From Fig. 3 and Fig. 4, we can obtain the following:

CONJECTURE 6. *For* $\lambda_2 < \lambda_2^c$ *and such* n_c *that there exists* l_c, *such* l_c *is a monotonically increasing function of* λ_2, *that is,* $\partial l_c / \partial \lambda_2 > 0$.

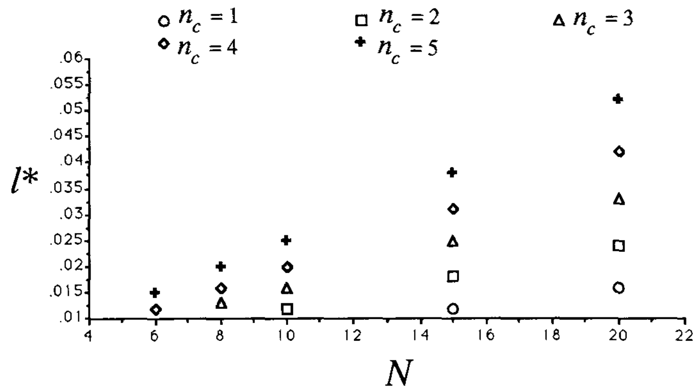

Fig. 5. N-l^* relation. $\lambda_2 = 0.1 \sim 1.0$.

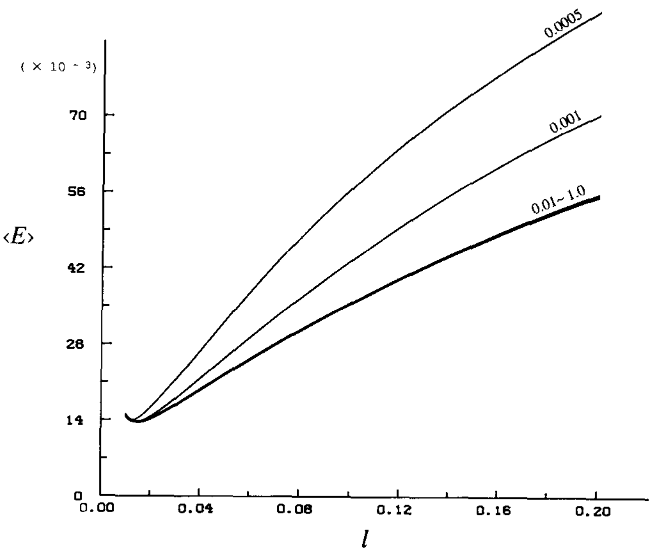

Fig. 6. l-$\langle E\rangle$ relation for some λ_2. $n_c = 5$. Attached value on each curve indicates λ_2.

CONJECTURE 7. *For $\lambda_2 < \lambda_2^c$ and such n_c that there does not exist l_c, the efficiency $\langle E\rangle$ is a monotonically increasing function of n_c, that is, $\partial\langle E\rangle/\partial n_c > 0$, independently of the value l.*

As for l^*, in this case, the similar conjecture to the above Conjecture 6 is obtained from Fig. 6:

CONJECTURE 8. *For $\lambda_2 < \lambda_2^c$, l^* is a monotonically increasing function λ_2, that is, $\partial l^*/\partial\lambda_2 > 0$.*

Further, from Fig. 6, we obtain the following conjecture about the efficiency $\langle E\rangle$:

CONJECTURE 9. *For $\lambda_2 < \lambda_2^c$, $\langle E\rangle$ is a monotonically decreasing function of λ_2, that is, $\partial\langle E\rangle/\partial\lambda_2 < 0$.*

Considering some cases such as those shown in Fig. 4, we deduce the following, too:

CONJECTURE 10. *For $\lambda_2 < \lambda_2^c$, the behavior-switching number n_c should be selected in the following way, in order to realize the higher searching efficiency, depending on the patch size l:*

(i) *large n_c* *when $l < l_c$*

(ii) *moderate n_c (> 1)* *when $l_c < l < l_\alpha$*

(iii) *small n_c* *when $l_\alpha < l$,*

where l_c is determined by such efficiency curves that have a common cross point at $l = l_c$, and l_α is determined by a cross point between the highest efficiency curve with l_c and the highest efficiency curve without l_c.

In Fig. 4, the selected n_c is more than 5 for the case (i), equal to 3 for (ii), equal to 2 for (iii). As a consequence, in case of Fig. 4, a two-mode searching is adaptable for $\lambda_2 < \lambda_2^c$, independently of l.

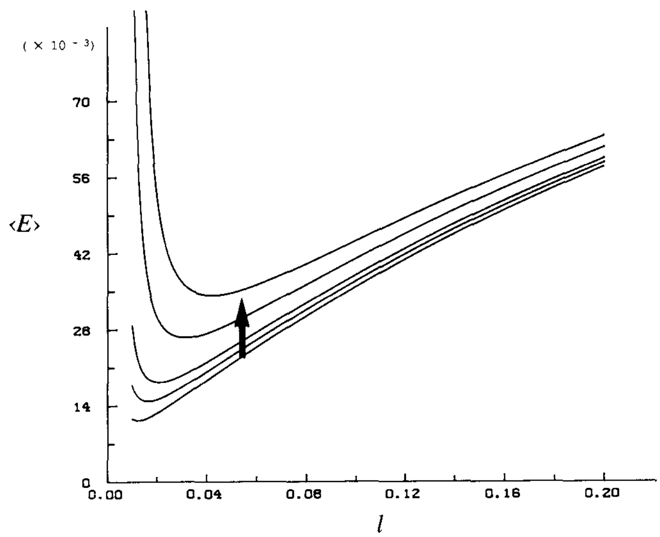

Fig. 7. l-$\langle E\rangle$ relation for some N. $n_c = 4$; $\lambda_2 = 0.1$. The curve monotonically increases with respect to N.

Incidentally, we notice that Fig. 7 shows that the efficiency is a monotonic increasing function of N. This is intuitively trivial because the searching efficiency should become higher when the total number of targets within a patch increases for fixed other parameters. In other words, for a fixed patch size, the larger the total number of targets within a patch, the smaller the distance between the nearest-neighber targets, in agreement with the modelling constraint (7). In such a situation, the searcher can easily find the target in the target-catching process. Then, for a constant $\langle n_1\rangle$, the searching efficiency of the target-catching process should become higher. This is the reason for the above-mentioned monotonicity.

3. Discussion

At first, we shall consider the case when there is no other constraints for the target's selection of its distribution way.

RESULT 1. *If there is no constraint on the distribution, the counter-behaving target takes l^* as its patch size at the goal of coevolutionary game with the searcher's searching behavior, while the cooperative-behaving target takes a dense patchy distribution (every nearest-neighbor targets touch each other in each patch) or a uniform distribution.*

In the coevolutionary game, the counter-behaving target's distribution is directed to reduce the searching efficiency $\langle E \rangle$. Thus, l^* is strategically adopted as the patch size optimal for the strategy of counter-behaving target's distribution. As for the case of cooperative-behaving target, the target tends to decrease its patch size if $l < l^*$ or to increase it if $l^* < l$, following Conjecture 1.

RESULT 2. *For the patch size $l < l_c$, the searcher takes a two-mode searching as its optimal strategy.*

This is a consequence of Conjectures 3 and 4. For the patch size l^*, the searching efficiency $\langle E \rangle$ is monotonically increasing with respect to n_c. Moreover, since l^* is monotonically increasing with respect to n_c, if the searcher takes a large n_c, the corresponding l^* must be relatively large. These results show that, at the coevolutionary goal, the counter-behaving target takes a special patch distribution against the searcher's two-mode searching and the cooperative-behaving target takes a uniform distribution against a simple-mode searching or a dense patchy distribution against a two-mode searching.

Next, we shall consider a special case when there is a lower bound of the patch size afforded to the target's distribution: for example, the case when the higher target's density leads to the smaller shear of resource (food etc.) or to the more serious intra-specific competition, so that very high density may violate the target's persistence. Hereafter we denote this minimal patch size $l_{\min}$. Besides, there may exist an upper bound of patch size $l_{\max}$. This situation is, for example, due to a limitation of available space for the target's distribution, or due to the mating efficiency among targets.

RESULT 3. *If $l_{\min} \leq l^* < l_{\max}$ then the counter-behaving target can take l^* as its patch size at the goal of coevolutionary game, while the cooperative-behaving target takes $l_{\min}$ or $l_{\max}$ at the goal. However, especially in case of the counter-behaving target, if $l^* < l_{\min}$, the coevolutionary game leads the patch size to $l_{\min}$. If $l_{\max} < l^*$, the game leads the patch size to $l_{\max}$.*

This follows from Conjecture 1. Further, by Conjectures 6 and 8, this result is plausible for the case when $\lambda_2 \ll \lambda_1$, that is, when the searcher takes an outstanding behavior change. It is shown that, when the available space for a patch is sufficiently small, the counter-behaving target strategically tends to take its lowest density within the patch because it must take its largest patch size at the coevolutionary goal. Note that, even if $l^* < l_{\min} < l_c$ at a moment of the game, it is likely that the counter-behaving target may take l^* as its patch size at the coevolutionary goal. This is because of the following reason: If $l^* < l_{\min} < l_c$, the target strategically tends to take $l_{\min}$. In this case, the searcher strategically tends to take a two-mode searching with a large n_c, in accordance with Conjecture 3. Since l^* is monotonically increasing with respect to n_c, l^* tends to become relatively large in the game, and it may occur for l^* to go beyond $l_{\min}$ at the coevolutionary goal.

On the other hand, focusing on the searcher's searching behavior, according to Conjectures 3 and 9, we obtain:

RESULT 4. *When $\lambda_2^c < \lambda_2$ (a moderate behavior change), if $l_c < l_{\min}$, a simple mode searching is strategically adaptable for the searcher. On the other hand, when $\lambda_2 < \lambda_2^c$ (an outstanding behavior change), if $l_\alpha < l_{\min}$, the searcher is likely to take strategically a simple mode searching. Otherwise, the searcher strategically takes a two-mode searching.*

Therefore, if the target cannot be strategically afforded to take a sufficiently small patch size, the coevolutionary goal of the searcher's searching behavior is likely to be a simple mode one, that is, to search without any change of its searching behavior. This is because a sufficiently large patch results in an easiness of patch-searching and a low efficient gain by the target-catching process due to a low density of targets within the patch. However, note that l_c monotonically increases with respect to N, that is, to the total number of targets in a patch, as mentioned in Conjecture 4. Thus, for a sufficiently large N, the condition $l_c < l_{\min}$ is likely to be violated. This leads to the following:

RESULT 5. *If the number of targets in a patch is sufficiently large, that is, if the target's density within the patch is sufficiently high, the searcher is likely to take a two-mode searching in the coevolutionary game.*

As mentioned in the analysis section, since the searching efficiency is a monotonically increasing function of N, the higher target's density within the patch is unfavorable for the counter-behaving target's distribution. In this sense, Result 5 seems to be not so appropriate for the counter-behaving target. But a high target's density within the patch is likely to occur for a reason (for instance, the mating efficiency among targets, etc.).

We notice that, because of Conjecture 2, these results may be very appropriate for the searcher which takes a relatively moderate two-mode searching (i.e., $\lambda_2^c < \lambda_2$). As for a sufficiently outstanding two-mode searching (i.e., $\lambda_2 < \lambda_2^c$), the two-mode searching is rather adaptable as the coevolutionary goal of the searching way, as mentioned in Conjectures 7 and 10:

RESULT 6. *A sufficiently outstanding two-mode searching is strategically rather adaptable.*

Moreover, from Conjecture 10 and Fig. 6, since the efficiency is higher with the sufficiently smaller λ_2 for a fixed behavior-switching number, the searcher tends to take strategically the smaller λ_2, that is, the more outstanding behavior change. By this reason, Result 6 is plausible. However, we must note that, since l_c and l^* are monotonically increasing with respect to $\lambda_2 < \lambda_2^c$, a simple mode searching may become adaptable for a sufficiently small λ_2 by the feature mentioned in Result 4. This seems to be because a too outstanding behavior change may reduce the searching efficiency due to the high search-missing probability in the target-catching process: Too small searching step distance (i.e., too small value of λ_2, in the sense of its expected value) increases the number of search-missing steps in the gap between targets.

Also within the framework of the present model, it is shown that a two-mode searching may be strategically adaptable for the searcher. Note that an appropriate small patch size can be strategically selected by the target. Thus, a patchy distribution of targets is very likely to be observed at the coevolutionary goal. Even if the patch size is subjected to a selective constraint, a two-mode searching can be adaptable as far as the possible minimal patch size is sufficiently small. This can be expressed as follows: As far as the target's density within a patch is sufficiently high, the searching gain becomes sufficiently large by changing the searching behavior after a patch is found. Two-mode searching is not likely to be adaptable for the patch of low target's density. Contrarily, if the patch size cannot be sufficiently small, the searcher strategically tends to take a simple mode searching. The reason seems to be the following: It is relatively easy for the searcher to find a patch in the space so that this easiness gives a sufficiently large gain to the searcher. Instead, the target-catching process cannot give the searcher a sufficient gain because of the time loss due to the relatively low target's density within the patch. In case of the counter-behaving target, only this case can cause a two-mode searching of the searcher as the coevolutionary goal.

Appendix.

In this appendix, making use of the probability density function given in each of two processes, we sketch the derivation of probabilities (4), (5), (16) and (17). Relations (14) and (15) are also proved.

$$\begin{aligned} P_1^{\mathrm{in}}(x) &= \sum_{m=0}^{+\infty} \Big[(1/2)\int_{\mathrm{I}} f_1(|x-y|_{\mathrm{clock}} + mA)dy \\ &\qquad + (1/2)\int_{\mathrm{I}} f_1(|x-y|_{\mathrm{anticlock}} + mA)dy\Big] \\ &= \sum_{m=0}^{+\infty} \Big[(1/2)\int_{\mathrm{I}} f_1(|x-y|_{\mathrm{clock}} + mA)dy \\ &\qquad + (1/2)\int_{\mathrm{I}} f_1(A-|x-y|_{\mathrm{clock}} + mA)dy\Big] \end{aligned}$$

$$= \sum_{m=0}^{+\infty} \Big[(1/2) \int_{\mathbf{I}} f_1(|x-y| + mA)dy$$

$$+ (1/2) \int_{\mathbf{I}} f_1(A - |x-y| + mA)dy\Big]$$

$$= (1/2\lambda_1) \sum_{m=0}^{+\infty} \exp(-mA/\lambda_1) \int_{\mathbf{I}} dy \big[\exp(-|x-y|/\lambda_1)$$

$$+ \exp(-A/\lambda_1) \exp(|x-y|/\lambda_1)\big]$$

$$= [(1/2\lambda_1)/\{1 - \exp(-A/\lambda_1)\}] \int_{\mathbf{I}} dy \big[\exp\{-(y-x)/\lambda_1\}$$

$$+ \exp(-A/\lambda_1) \exp\{(y-x)/\lambda_1\}\big]$$

$$= \sinh(l/2\lambda_1) \cosh\{(2x - A + l)/2\lambda_1\} \mathrm{sech}(A/2\lambda_1)$$

where $|x-y|_{\mathrm{clock}}(|x-y|_{\mathrm{anticlock}})$ denotes the distance measured in the clockwise (the anticlockwise) direction between x and y, and where $|x-y|$ denotes the absolute value of the difference between coordinates of x and y on $\mathbf{S}^1$. For the following calculation for the target-catching process, the notations have the same meanings.

$$P_1{}^g(x \to y) = \sum_{m=0}^{+\infty} \big[(1/2) f_1(|x-y|_{\mathrm{clock}} + mA) + (1/2) f_1(|x-y|_{\mathrm{anticlock}} + mA)\big]$$

$$= (1/2) \sum_{m=0}^{+\infty} \big[f_1(|x-y|_{\mathrm{clock}} + mA) + f_1(A - |x-y|_{\mathrm{clock}} + mA)\big]$$

$$= (1/2\lambda_1) \sum_{m=0}^{+\infty} \big[\exp\{-(|x-y| + mA)/\lambda_1\}$$

$$+ \exp\{-(A - |x-y| + mA)/\lambda_1\}\big]$$

$$= (1/2\lambda_1) \big[\exp(-|x-y|/\lambda_1)$$

$$+ \exp\{-(A - |x-y|)/\lambda_1\}\big] / \big[1 - \exp(-A/\lambda_1)\big]$$

$$= (1/2\lambda_1) \cosh\{(A - 2|x-y|)/2\lambda_1\} \mathrm{sech}(A/2\lambda_1)$$

$$P_2{}^{\mathrm{in}}(z) = \sum_{m=0}^{+\infty} \Big[(1/2) \int_{\mathbf{i}} f_2(|z-y|_{\mathrm{clock}} + md)dy$$

$$+ (1/2) \int_{\mathbf{i}} f_2(|z-y|_{\mathrm{anticlock}} + md)dy\Big]$$

$$= \sum_{m=0}^{+\infty} \Big[(1/2) \int_{\mathbf{i}_1 \cup \mathbf{i}_2} f_2(|z-y|_{\mathrm{clock}} + md)dy$$

$$+ (1/2)\int_{\mathrm{i}_1\cup \mathrm{i}_2} f_2(|z-y|_{\text{anticlock}} + md)dy\big]$$

$$= \sum_{m=0}^{+\infty} \big[(1/2)\int_{\mathrm{i}_1} f_2(d-|z-y|_{\text{anticlock}} + md)dy$$

$$+ (1/2)\int_{\mathrm{i}_1} f_2(|z-y|_{\text{anticlock}} + md)dy\big]$$

$$+ \big[(1/2)\int_{\mathrm{i}_2} f_2(|z-y|_{\text{clock}} + md)dy$$

$$+ (1/2)\int_{\mathrm{i}_2} f_2(d-|z-y|_{\text{clock}} + md)dy\big]$$

$$= \sum_{m=0}^{+\infty} \big[(1/2)\int_{\mathrm{i}_1} f_2(d-z+y+md)dy + (1/2)\int_{\mathrm{i}_1} f_2(z-y+md)dy\big]$$

$$+ \big[(1/2)\int_{\mathrm{i}_2} f_2(y-z+md)dy + (1/2)\int_{\mathrm{i}_2} f_2(d-y+z+md)dy\big]$$

$$= \sum_{m=0}^{+\infty} \big[(1/2)\int_{\mathrm{i}_1} f_2(d-z+y+md)dy + (1/2)\int_{\mathrm{i}_1} f_2(z-y+md)dy\big]$$

$$+ \big[(1/2)\int_{\mathrm{i}_1} f_2(d-r+y'-z+md)dy'$$

$$+ (1/2)\int_{\mathrm{i}_1} f_2(r-y'+z+md)dy'\big]$$

$$= \sum_{m=0}^{+\infty} (1/2)\int_{\mathrm{i}_1} dy\big[f_2(d-z+y+md) + f_2(z-y+md)$$

$$+ f_2(d-r+y-z+ml) + f_2(r-y+z+md)\big]$$

$$= (1/2\lambda_2)\sum_{m=0}^{+\infty} \exp(-md/\lambda_2)\int_{\mathrm{i}_1} dy\big[\exp\{-(d-z)/\lambda_2\}\exp(\neg y/\lambda_2)$$

$$+ \exp(-z/\lambda_2)\exp(y/\lambda_2) + \exp\{-(d-r-z)/\lambda_2\}\exp(-y/\lambda_2)$$

$$+ \exp\{-(r+z)/\lambda_2\}\exp(y/\lambda_2)\big]$$

$$= \big[(1/2)/\{1-\exp(-d/\lambda_2)\}\big]\big[\exp\{-(d-z)/\lambda_2\}\{1-\exp(-r/\lambda_2)\}$$

$$+ \exp(-z/\lambda_2)\{\exp(r/\lambda_2)-1\}$$

$$+ \exp\{-(d-r-z)/\lambda_2\}\{1-\exp(-r/\lambda_2)\}$$

$$+ \exp\{-(r+z)/\lambda_2\}\{\exp(r/\lambda_2)-1\}\big]$$

$$= \operatorname{sech}(d/2\lambda_2)\sinh(r/\lambda_2)\cosh\{(2z-d)/2\lambda_2\}$$

$$
\begin{aligned}
P_2{}^{\mathrm{out}}(z) &= \sum_{m=0}^{+\infty} \{f_2(z+md) + f_2(d-z+md)\} \\
&= (1/\lambda_2)\big[\exp(-z/\lambda_2) + \exp\{(z-d)/\lambda_2\}\big] \sum_{m=0}^{+\infty} \exp(-md/\lambda_2) \\
&= (1/\lambda_2)\big[\exp(-z/\lambda_2) + \exp\{(z-d)/\lambda_2\}\big] / \big[1 - \exp(-d/\lambda_2)\big] \\
&= (1/\lambda_2)\cosh\{(2z-d)/2\lambda_2\}\mathrm{sech}(-d/2\lambda_2)
\end{aligned}
$$

$$
\begin{aligned}
\langle M \rangle &= \sum_{m=0}^{+\infty} \sum_{\substack{n_j=1 \\ (j=1,2,\ldots,m-1)}}^{n_c} m \prod_{k=1}^{m-1} P_{\langle z \rangle}^{n_k} q^c{}_{\langle z \rangle} \\
&= \sum_{m=1}^{+\infty} m \cdot q^c{}_{\langle z \rangle} \cdot (P^c{}_{\langle z \rangle})^{m-1} = 1/(1 - P^c{}_{\langle z \rangle}).
\end{aligned}
$$

Here, for mathematical convenience to calculate $\langle n_2 \rangle$ we shall begin by calculating the expected total number, say $\langle n_2 \rangle_M$, of steps for the searcher to catch M targets and give up the target-catching process after n_c steps, averaged with respect to the configuration of searching, where the number of steps to catch another target after one must be equal to or less than the giving-up step n_c.

$$
\begin{aligned}
\langle n_2 \rangle_M &= \sum_{\substack{n_j=1 \\ (j=1,2,\ldots,M-1)}}^{n_c} (n_1 + n_2 + \ldots + n_{M-1} + n_c) \prod_{k=1}^{m-1} P_{\langle z \rangle}^{n_k} q^c{}_{\langle z \rangle} \\
&= (M-1)(P^c{}_{\langle z \rangle})^{M-2} \cdot q^c{}_{\langle z \rangle} \cdot \sum_{n=1}^{n_c} n \cdot P^n{}_{\langle z \rangle} + (P^c{}_{\langle z \rangle})^{M-1} \cdot n_c \cdot q^c{}_{\langle z \rangle} \\
&= (M-1)(P^c{}_{\langle z \rangle})^{M-1} \cdot q^c{}_{\langle z \rangle} \cdot \langle n_2 \le n_c \rangle + (P^c{}_{\langle z \rangle})^{M-1} \cdot n_c \cdot q^c{}_{\langle z \rangle} \\
&= (P^c{}_{\langle z \rangle})^{M-1} \cdot q^c{}_{\langle z \rangle} \{(M-1)\langle n_2 \le n_c \rangle + n_c\}.
\end{aligned}
$$

Then, the required expected total number of steps in the target-catching process before the searcher gives it up is:

$$
\begin{aligned}
\langle n_2 \rangle &= \sum_{M=1}^{+\infty} \langle n_2 \rangle_M \\
&= \sum_{M=1}^{+\infty} (P^c{}_{\langle z \rangle})^{M-1} \cdot q^c{}_{\langle z \rangle} \{(M-1)\langle n_2 \le n_c \rangle + n_c\}
\end{aligned}
$$

$$= q^c{}_{\langle \mathbf{z} \rangle} \langle n_2 \le n_c \rangle \cdot \left\{ \sum_{M=1}^{+\infty} (M-1)(P^c{}_{\langle \mathbf{z} \rangle})^{M-1} \right\} + n_c \cdot q^c{}_{\langle \mathbf{z} \rangle} / (1 - P^c{}_{\langle \mathbf{z} \rangle})$$

$$= q^c{}_{\langle \mathbf{z} \rangle} \langle n_2 \le n_c \rangle \cdot P^c{}_{\langle \mathbf{z} \rangle} / (1 - P^c{}_{\langle \mathbf{z} \rangle})^2 + n_c$$

$$= \{ P^c{}_{\langle \mathbf{z} \rangle} / (1 - P^c{}_{\langle \mathbf{z} \rangle}) \} \langle n_2 \le n_c \rangle + n_c$$

$$= \{ \langle M \rangle - 1 \} \langle n_2 \le n_c \rangle + n_c.$$

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