

Flow-Radius Relation of Symmetrical Dichotomous Vascular System from Minimum Work Principle

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Abstract. We set up a mathematical modelling of dichotomous vascular system and derive a relation required for the structural optimality based on the minimum work principle. It is discussed how the exponent n in the relation $Q \propto r^n$ between the flow rate Q and the vessel radius r reflects physiologic and structural characteristics of the vascular system. It is shown that the introduction of the fractal nature of the vascular system provides a principle to explain and study the observed variety of the exponent n .

1. Introduction

MURRAY (1926) presented a mathematical model on the blood flow through the vessel, based on the minimum work principle. He considered the structural characteristics of the vascular system to minimize the energy consumption of the flow in the vessel. He assumed the Hagen-Poiseuille flow with the flow rate,

$$Q = -\frac{\pi r^4}{8\eta} \frac{\partial p}{\partial x}, \quad (1)$$

where r is the mean radius of the vessel, η the viscosity of the fluid and $\partial p/\partial x$ indicates the pressure gradient (Fig. 1). The Hagen-Poiseuille relation (1) is applicable only for laminar flows (REYNOLDS, 1883).

From the Hagen-Poiseuille relation (1), the pressure difference Δp between two edges of the vessel of length l can be obtained as follows:

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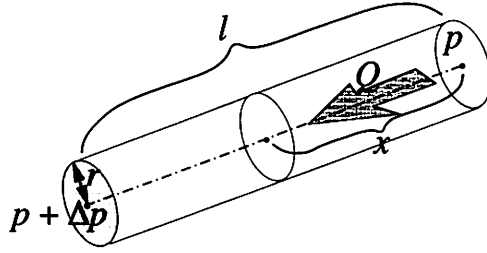


Fig. 1. Flow through the vessel considered by MURRAY (1926). For detailed explanation, see the main text.

$$\Delta p = \frac{8\eta l Q}{\pi r^4}. \quad (2)$$

Accordingly, the kinetic energy consumption E_1 in the Hagen-Poiseuille flow in the vessel of length l is given by

$$E_1 = Q \cdot \Delta p = \frac{8\eta l}{\pi r^4} Q^2. \quad (3)$$

In addition, the flow consumes a portion of its energy due to the physiologic function. In MURRAY (1926), this physiologic energy consumption E_2 is assumed proportional to the volume occupied by the vessel itself:

$$E_2 = b \cdot V = b\pi r^2 l, \quad (4)$$

where the volume is assumed to be that of a smooth circular tube with the radius r and the length l . We suppose that the structure which minimizes the total energy consumption $E = E_1 + E_2$ would be optimal for the tissue, and that the structural factors are represented only by the radius r and the length l . Then, we obtain a relation between Q and r in the following way:

$$\frac{\partial E}{\partial r} = -\frac{32\eta}{\pi r^5} Q^2 l + 2b\pi r l = 0, \quad (5)$$

hence

$$Q = \frac{\pi}{4} \sqrt{\frac{b}{\eta}} \cdot r^3. \quad (6)$$

This " $Q \propto r^3$ " relation can be derived by another mathematical modelling with an assumption that the vascular system has such a structure that the share stress on the internal wall of the vessel is constant, i.e., the vessel wall is assumed sensitive to the flow rate Q , and controls the radius r to keep τ constant. For Hagen-Poiseuille flow, the share tension τ is given by $\tau = 4\eta Q/\pi r^3 \propto Q/r^3$, hence $Q \propto r^3$. It is to be noted that these models are based on the local optimality of the vascular system and do not take the global structure of the system, specifically the branching structure, into account.

WOLINSKY and GLAGOV (1967) examined the relation between the body weight and the radius of aorta for some samples from a variety of animals, and found that the body weight was approximately proportional to the cube of the radius of aorta. If the volume occupied by the aorta is proportional to the cube of the radius of aorta, their result suggests that the blood flow rate, which must balance with the need proportional to the weight, is proportional to the cube of the radius of aortal vessel. Their work has been frequently referred as the " $Q \propto r^3$ " relation.

Physiologic researches have shown that the data concerning the flow rate and the radius of the vessel present a relation $Q \propto r^n$ with $n < 3$. For several vessels of a variety of organ tissues, SUWA *et al.* (1963) estimated n as 2.4~2.8 (see also SUWA and TAKAHASHI, 1971). GROAT (1948) reported that $n \sim 2.6$. MANDELBROT (1982) called the exponent n *the diameter exponent or the para dimension*. THOMA (1901, 1920) tried to derive n theoretically, and gave its value less than 2.4. However, the THOMA's estimation is too small compared with the observed values of n .

After all, we do not have a sufficient model to elucidate the $Q \propto r^n$ relation, and there is a growing need for an improvement of mathematical modelling.

In this paper, based on the minimum work principle, a mathematical modelling on the optimal construction of the dichotomous branching vascular system is analyzed. It will be shown that the introduction of the fractal nature of the vascular system into the model provides a new insight into the exponent n in the $Q \propto r^n$ relation.

2. Energy Consumption Proportional to Vessel Surface

In MURRAY's modelling, the physiologic energy consumption E_2 is assumed proportional to the volume occupied by the vessel itself. Whereas, it is possible to consider that the consumption E_2 would be proportional to the surface area S of the vessel. This hypothesis is based on the idea that the physiologic energy consumption might be due to the interaction between the internal and the external of the vessel wall, for instance, due to exchanging materials at the surface.

Both of the internal and the external vessel walls are not smooth and have a fine and complex structure. The vessel radius r in MURRAY's modelling can be regarded as a mean value to conserve the vessel volume. In the case where the surface area is essential, we should take the morphology of the wall into account. Even though the volume of the vessel can be given well by $\pi r^2 l$ with the mean radius r and the longitudinal distance l between two edges of the vessel, the surface area of the vessel, the surface area of the vessel may be more than $2\pi r l$ due to the complex structure (Fig. 2). Applying the concept of *fractal*

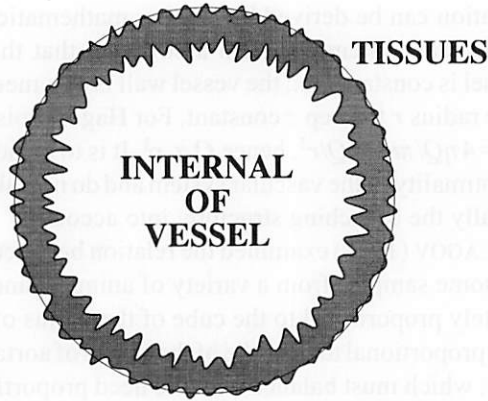


Fig. 2. Periphery of the cross-section of vessel with a fine structure of the fractal dimension d .

by MANDELBROT (1982), we can consider the effective surface area proportional to $r^d l$ instead of rl , where d denotes the fractal dimension of the periphery of the cross-section of the vessel. The case with $d = 1$ is for the completely smooth vessel wall corresponding to a circular tube, while the complex surface structure of the vessel wall may have d with $1 < d < 2$. It is evident that the case $d = 2$ gives mathematically the same modelling as MURRAY's one.

Now, the physiologic energy consumption E_2 proportional to the surface area S is given by:

$$E_2 = \beta \cdot S = 2\pi\beta r^d l, \quad (7)$$

where β is a positive constant. Therefore, by the minimum work principle $\partial E / \partial r = \partial [E_1 + E_2] / \partial r = 0$ with E_1 and E_2 given by (3) and (7), respectively, we can easily obtain the expression:

$$Q = \frac{\pi}{4} \sqrt{d \cdot \frac{\beta}{\eta}} \cdot r^{2+d/2}. \quad (8)$$

Therefore, we can obtain the " $Q \propto r^n$ " relation with

$$n = 2 + \frac{d}{2}, \quad (9)$$

where $2.5 \leq n \leq 3.0$, since $1 \leq d \leq 2$. Note that the case with $n < 2.5$ cannot be predicted by our model. The values $n = 2.5 \sim 2.8$ estimated by SUWA *et al.* (1963) correspond to $d =$

1.0~1.6. Thus, the variety of the values of exponent n might be due to the variety of the fractal dimensions of the vessel wall.

3. Symmetric Geometrical Dichotomous Branching System

Now, we consider a symmetric dichotomous branching vascular system with a geometrical structure as shown in Fig. 3. Two daughter branches have common radius and length. The radius is assumed to reduce with a constant ratio at each branching:

$$r_{k+1} = \rho r_k, \quad (10)$$

where r_k is the radius of the k -th branch, and ρ is a positive constant less than 1. On the other hand, the flow is assumed to be divided equally to the two daughter branches, i.e.,

$$Q_{k+1} = \frac{Q_k}{2}, \quad (11)$$

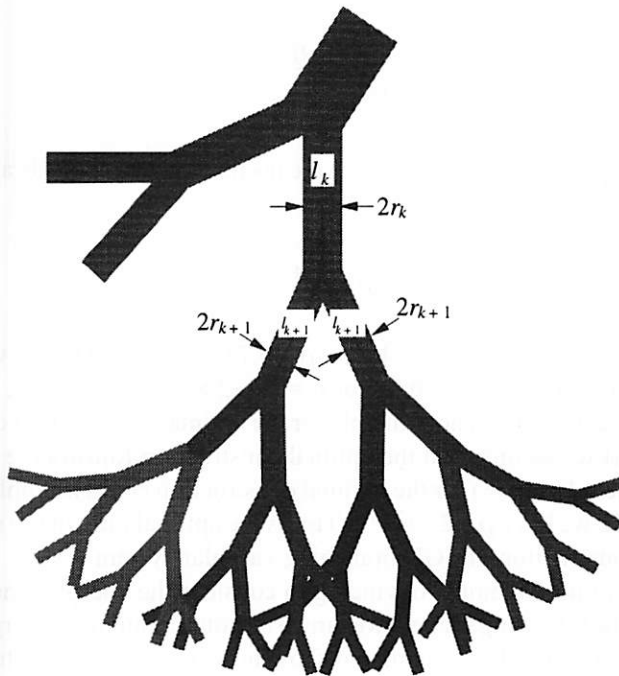


Fig. 3. Symmetric geometrical dichotomous (SGD) branching vascular system. $r_{k+1} = \rho r_k$ and $l_{k+1} = \mu l_k$ where r_k and l_k are respectively the radius and the length of the k -th branch, and ρ and μ are constant less than 1.

where Q_k is the flow rate in the k -th branch. The 0-th branch is arbitrarily and appropriately selected, for instance, depending on which tissues its physiological function contributes to.

As for the length of the branch, from a sampling study of a variety of organ vessels, SUWA *et al.* (1963) reported a proportional relation between the branch length l and a power of the radius r , i.e., $l \propto r^i$. The exponent i was estimated as $i = 0.8 \sim 1.2$ from the data. From this relation and (10), we can easily derive a reduction of the branch length

$$l_{k+1} = \rho^i l_k, \quad (12)$$

where l_k is the length of the k -th branch. This relation means that the branch length also reduces geometrically with a constant ratio $\mu = \rho^i < 1$.

In this paper, we consider this symmetric geometrical dichotomous (SGD) branching vascular system with the reduction ratios ρ for the radius and $\mu = \rho^i$ for the length. Note that, for our SGD vascular system, no assumption is made for the branching angle, and that the SGD vascular system does not necessarily have the self-similarity nature.

It should be remarked that the $Q \propto r^n$ relation is always established for our SGD branching vascular system, because of the relations (10) and (11):

$$\frac{Q_k}{Q_0} = \left(\frac{r_k}{r_0} \right)^\gamma, \quad (13)$$

where $\gamma = -\log 2 / \log \rho > 0$. This relation (13) means that $\gamma = n$ for SGD branching vascular system, and

$$\rho = 2^{-1/n}. \quad (14)$$

For $n = 3.0$, $\rho = 0.79$, while, for $n = 2.4 \sim 2.8$, $\rho = 0.75 \sim 0.78$. In addition, with the relation $\mu = \rho^i$, μ is estimated as $\mu = 0.73 \sim 0.81$ for $n = 2.4 \sim 2.8$ and $i = 0.8 \sim 1.2$.

So far we have introduced no principle for the optimal construction of the branching vascular system. If we assume that the optimal construction follows the *local* minimum work principle, the relation (8) for the optimal construction could be applied. Then, from $n = 2 + d/2$ and (14), we have $\rho = 2^{-1/(2+d/2)}$. This is the optimal ratio of the radius reduction for the optimal construction of SGD branching vascular system.

However, it would be more convincing to consider the energy consumption in the whole system. In the following section, we turn to the total cumulative energy consumption over SGD branching vascular system, and show how the optimal construction is determined by minimizing it.

4. Total Cumulative Energy Consumption

The kinetic energy consumption $e_j^{(1)}$ for the flow rate Q_j in the j -th branch with the length l_j and the radius r_j is given in analogy to (3) by

$$e_j^{(1)} = Q_j \cdot \Delta p_j = \frac{8\eta}{\pi} \cdot \frac{Q_j^2 l_j}{r_j^4} = \frac{8\eta Q_0^2 l_0}{\pi r_0^4} \cdot \left(\frac{\rho^{i-4}}{4}\right)^j, \quad (15)$$

where we used the relations (10), (11), and (12). Since the total number of the k -th branches of SGD vascular system is 2^k , the cumulative kinetic energy consumption $E_k^{(1)}$ over the 0-th ~ the k -th branches is given as follows:

$$E_k^{(1)} = \sum_{j=0}^k 2^j \cdot e_j^{(1)} = \frac{8\eta Q_0^2 l_0}{\pi r_0^4} \sum_{j=0}^k \left(\frac{\rho^{i-4}}{2}\right)^j. \quad (16)$$

On the other hand, the surface area for one of the k -th branches is now assumed to be given by $2\pi r_k^d l_k$. Provided that the physiologic energy consumption is proportional to the vessel surface area, the cumulative physiologic energy consumption $E_k^{(2)}$ over the 0-th ~ the k -th branches is

$$E_k^{(2)} = \sum_{j=0}^k 2^j \cdot (\beta \cdot 2\pi r_j^d l_j) = 2\pi\beta r_0^d l_0 \sum_{j=0}^k (2\rho^{i+d})^j. \quad (17)$$

Now, by differentiating $E_k^{(1)} + E_k^{(2)}$ with respect to ρ and using relation among parameters, we can obtain the following results (see Appendix):

$$n = 2 + \frac{d}{2}, \quad (18)$$

$$Q_k = \frac{\pi}{2} \sqrt{\frac{d+i}{4-i}} \cdot \frac{\beta}{\eta} \cdot r_k^{2+d/2} \quad (k = 0, 1, 2, \dots), \quad (19)$$

$$i < 4. \quad (20)$$

The condition (20) is necessary for the existence of the optimal ρ . Equation (18) coincides with the result by the local minimum work principle. The result (19) is different from (8)

by the constant factor. Only when $i = 0$, that is, only when every length of branches is equal, they are coincides with each other. Since $i = 0.8 \sim 1.2$ by SUWA *et al.* (1963), condition (20) seems satisfied in general for real vascular systems.

With these results, the total consumption $E_k^{(1)} + E_k^{(2)}$ becomes

$$E_k^{(1)} + E_k^{(2)} = 2\pi\beta r_0^d l_0 \cdot \frac{4+d}{4-i} \cdot \sum_{j=0}^k 2^{\nu_j}, \quad (21)$$

where

$$\nu = \frac{4 - (2 + d/2 + i)}{2 + d/2} = \frac{4 - (n + i)}{n}. \quad (22)$$

According to SUWA *et al.* (1963), the estimated values of $n + i$ were always less than 4.0, hence $\nu > 0$, for several vessels of organ tissues. Applying this result to (22), we can say that the total cumulative energy consumption for SGD vascular system tends to diverge infinitely as the branch generation gets larger.

5. Conclusion

From our result $n = 2 + d/2$, with $n = 2.4 \sim 2.8$ by SUWA *et al.* (1963), we can conclude that $d = 1.0 \sim 1.6$. Therefore, the inner wall of the vessel is expected to have a fine structure with the fractal dimension between 1.0 and 1.6.

SUWA *et al.* (1963) reported that n is larger in the structure with $r < 100 \mu$ than that with $r \geq 100 \mu$. SUWA *et al.* (1963) observed also that the ratio of the thickness of vessel wall to its peripheral length significantly changed, so that the wall is significantly thick relative to the peripheral length for the narrower vessel. This result may imply the difference of the fine structure of the vessel walls. If the wall structure has the higher fractal dimension d in the vessel with $r < 100 \mu$ than that with $r \geq 100 \mu$, the value of n is expected to be larger for the narrower structure. Thus, the variation of d is a possible explanation for the variation of n in the framework of our modelling.

Another possible explanation for the increasing value of n as the radius r gets smaller is the difference of the fractal dimensions D_f of the branching structure. MATSUO *et al.* (1990) examined cat brain vessels and human retinal ones, and observed that $D_f = 1.7 \sim 1.8$ for the large scale and $D_f \sim 1.3$ for the small scale. They estimated the cross over scale at around 140μ . In the present modelling, such fractal nature of branching structure for the vascular system is not introduced. With an improved mathematical modelling, it will be possible to show the relations among the exponents n and i , and the fractal natures of the branching vascular system.

Lastly, it is conjectured that the larger value of n for the structure of $r < 100 \mu$ may be explained by the difference of both of the branching structure and the fine structure of

vessel surface.

In this paper, we demonstrate that the fractal structure of the inner wall surface of the vascular system combined with the optimal principle leads to predictions of the real construction of the organ system and its morphology. We hope that this work will give some stimuli to the researches for understanding the real physiologic function.

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APPENDIX

In this appendix, it is shown how the derivative of $E_k^{(1)} + E_k^{(2)}$ with respect to ρ leads to the relation among parameters.

From (16) and (17), the total cumulative energy consumption over the 0-th ~ the k -th branches is given by

$$\begin{aligned} E_k^{(1)} + E_k^{(2)} &= \frac{8\eta Q_0^2 l_0}{\pi r_0^4} \sum_{j=0}^k \left(\frac{\rho^{i-4}}{2} \right)^j + 2\pi\beta r_0^d l_0 \sum_{j=0}^k (2\rho^{i+d})^j \\ &= \frac{8\eta Q_0^2 l_0}{\pi r_0^4} \sum_{j=0}^k \left\{ \left(\frac{\rho^{i-4}}{2} \right)^j + \frac{\pi^2 \beta r_0^{4+d}}{4\eta Q_0^2} \cdot (2\rho^{i+d})^j \right\}. \end{aligned} \quad (A1)$$

Hence,

$$\frac{\partial [E_k^{(1)} + E_k^{(2)}]}{\partial \rho} = \frac{8\eta Q_0^2 l_0}{\pi r_0^4} \sum_{j=0}^k j \cdot \left\{ \frac{i-4}{\rho} \cdot \left(\frac{\rho^{i-4}}{2} \right)^j + \frac{\pi^2 \beta r_0^{4+d}}{4\eta Q_0^2} \cdot \frac{i+d}{\rho} \cdot (2\rho^{i+d})^j \right\}. \quad (A2)$$

The right-hand side of (A2) vanishes only when the following condition is satisfied for any j :

$$\frac{i-4}{\rho} \cdot \left(\frac{\rho^{i-4}}{2} \right)^j + \frac{\pi^2 \beta r_0^{4+d}}{4\eta Q_0^2} \cdot \frac{i+d}{\rho} \cdot (2\rho^{i+d})^j = 0. \quad (A3)$$

For the existence of the value of ρ for any j , it is necessary from (A3) that

$$i < 4, \quad (A4)$$

and

$$\frac{\rho^{i-4}}{2} = 2\rho^{i+d},$$

that is,

$$\rho = 2^{-1/(2+d/2)}. \quad (\text{A5})$$

This means that $n = 2 + d/2$. With (A4) and (A5), the relation (A3) gives

$$Q_0 = \frac{\pi}{2} \sqrt{\frac{d+i}{4-i}} \cdot \frac{\beta}{\eta} \cdot r_0^{2+d/2}. \quad (\text{A6})$$

By means of the relations (10) and (11), the result (A6) gives (19).

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