A strengthening of the Assmus–Mattson theorem based on the displacement and split decompositions

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Outline

1. Introduction
2. Discussions
3. Remarks

Motivation/Background

Bose–Mesner algebra (1959)

- commutative
- codes and designs (Delsarte, 1973)
- LP bound

Terwilliger algebra (1992)

- non-commutative
- more information
- SDP bound (Schrijver, 2005)

Extend the Delsarte theory based on the Terwilliger algebra!!

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Basic notations from coding theory

In this talk, we consider codes of length D over $\mathbb{F}_q$.

- $\mathbf{0} := (0, 0, \ldots, 0)$: the zero vector
- $\partial(x, y) := |\{1 \leqslant i \leqslant D : x_i \neq y_i\}|$: the Hamming distance
- $\mathrm{supp}(x) = \{1 \leqslant i \leqslant D : x_i \neq 0\}$: the support of x
- $\mathrm{wt}(x) := \partial(x, \mathbf{0}) = |\mathrm{supp}(x)|$: the Hamming weight of x

The Bose–Mesner algebra (of the Hamming scheme)

- $A \in \mathrm{Mat}_{\mathbb{F}_q^D}(\mathbb{C})$: the adjacency matrix:

$$A_{xy} := \begin{cases} 1 & \text{if } \partial(x,y) = 1 \\ 0 & \text{otherwise} \end{cases}$$

- $M := \mathbb{C}[A] \subseteq \mathrm{Mat}_{\mathbb{F}_q^D}(\mathbb{C})$: the Bose–Mesner algebra
- $E_0, E_1, \ldots, E_D$: the primitive idempotents of M

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The Terwilliger algebra

- $E_0^*, E_1^*, \ldots, E_D^* \in \mathrm{Mat}_{\mathbb{F}_q^D}(\mathbb{C})$: the *dual idempotents*:

$$(E_i^*)_{xy} = \begin{cases} 1 & \text{if } \mathrm{wt}(x) = i,\ x = y \\ 0 & \text{otherwise} \end{cases}$$

- $T := \mathbb{C}[A, E_0^*, E_1^*, \ldots, E_D^*]$: the *Terwilliger algebra*
- $T \curvearrowright V := \mathrm{Span}_{\mathbb{C}}\{\hat{x} : x \in \mathbb{F}_q^D\}$
- V, $\langle\ ,\ \rangle$: the *standard T-module* ($\langle \hat{x}, \hat{y} \rangle := \delta_{xy}$)
- $M\hat{\mathbf{0}}$: the *primary T-module*

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The Assmus–Mattson Theorem (1969)

- $C \subseteq \mathbb{F}_q^D$: a linear code with minimum weight δ
- $C^\perp \subseteq \mathbb{F}_q^D$: the dual code of C, with minimum weight δ^*
- $\chi_C \in V$: the characteristic vector of C:

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Assumption (algebraic):

Suppose $t \in \{1, 2, \ldots, D\}$ is such that at least one of the following holds:

- $|\{1 \leqslant i \leqslant D - t : E_i\chi_C \neq 0\}| \leqslant \delta - t$
- $|\{1 \leqslant i \leqslant D - t : E_i^*\chi_C \neq 0\}| \leqslant \delta^* - t$

Conclusion (combinatorial):

For every $0 \leqslant k \leqslant D$,

$$\{\mathrm{supp}(x) : x \in C,\ \mathrm{wt}(x) = k\} \text{ (counting repeats)}$$

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The goal of this talk

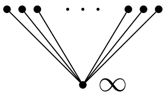
Interpret the Assmus–Mattson theorem in terms of the irreducible T-modules!

(Algebraic property of C) $\longleftrightarrow$ (Combinatorial property of C)

Outline

The Hamming lattice (Delsarte, 1976)

- $\mathbb{F}_q \cup \{\infty\}$: the "claw semilattice" of order $q+1$



- $(\mathscr{L}, \preccurlyeq)$: the direct product of D claw semilattices:
  - $\mathscr{L} = (\mathbb{F}_q \cup \{\infty\})^D$
  - $u \preccurlyeq v \iff u_i = \infty$ or $u_i = v_i$ $(1 \leqslant i \leqslant D)$

Remark

$(\mathscr{L}, \preccurlyeq)$ is ranked: $\mathrm{rank}(u) = |\{i : u_i \neq \infty\}|$

Remark

$\mathbb{F}_q^D$ forms the top fibre of $(\mathscr{L}, \preccurlyeq)$.

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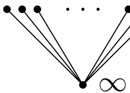

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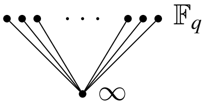

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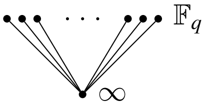

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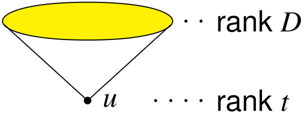

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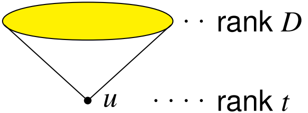

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Restatement of the conclusion (I)

The following are equivalent:

(a) $\{\text{supp}(x) : x \in C,\ \text{wt}(x) = k\}$: a t-design.

(b) $\{\{1, 2, \ldots, D\} - \text{supp}(x) : x \in C,\ \text{wt}(x) = k\}$: a t-design.

(c) $\langle E_k^* \chi_C, \chi_{\succcurlyeq u} \rangle$ is independent of $u \preccurlyeq \mathbf{0}$ with rank t.

$(\because)$
$$(\underbrace{0, 0, \ldots, 0}_{t}, \infty, \infty, \ldots, \infty) \preccurlyeq \mathbf{0}$$

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Facts on irreducible T-modules

- W : an irreducible T-module in V
- $r := \min\{i : E_i^* W \neq 0\}$: the *endpoint* of W
- $d := |\{i : E_i^* W \neq 0\}| - 1$: the *diameter* of W

Remark

$$W = E_r^* W \perp E_{r+1}^* W \perp \cdots \perp E_{r+d}^* W$$
$$= E_r W \perp E_{r+1} W \perp \cdots \perp E_{r+d} W$$

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The space $\triangle$ (Terwilliger, 2005)

- Caughman (1999) showed $2r + d \geqslant D$.
- $\eta := 2r + d - D$: the *displacement* of W $\geqslant 0$
- $\triangle$: the linear span of the irreducible T-modules W with displacement 0

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The structure of Δ

Lemma (Ito-Tanabe-Terwilliger, 2001)

$$\Delta = \sum_{i=0}^{D} \big((E_0^* V + \cdots + E_{D-i}^* V) \cap (E_0 V + \cdots + E_i V) \big) \text{ (direct sum).}$$

- $u \preccurlyeq \mathbf{0}$: rank t

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$$\chi_{\succcurlyeq u} \in (E_0^* V + \cdots + E_{D-t}^* V) \cap (E_0 V + \cdots + E_t V).$$

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Restatement of the conclusion (II)

(c) $\langle E_k^* \chi_C, \chi_{\succcurlyeq u} \rangle$ is independent of $u \preccurlyeq \mathbf{0}$ with rank t.

Lemma

Condition (c) *is implied by the following:*

(d) χ_C *is orthogonal to every irreducible* T*-module* W *with* $\eta = 0$ *and* $1 \leqslant r \leqslant t$.

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Lemma

Condition (c) *is implied by the following:*

(d) χ_C *is orthogonal to every irreducible* T*-module* W *with* $\eta = 0$ *and* $1 \leqslant r \leqslant t$.

(Algebraic property of C) $\longleftrightarrow$ (Combinatorial property of C)

$\triangle$

Outline

Remarks

- The Assmus–Mattson theorem is valid for nonlinear codes as well.

Example

The $[12, 6, 6]$ extended ternary Golay code is self-dual and has weight distribution

$$(1, \underbrace{0, \overset{\delta=3}{0}, 0, 0, 0, 264, 0, 0, 440, 0, 0,}_{\#\text{(dual)weights}=2} \underset{t=1}{24}).$$

Thus each coset of weight 3 (i.e., $\delta = 3$) supports 1-designs since $2 \leqslant 3 - 1$.

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- The Assmus–Mattson theorem can be generalized to other P-& Q-polynomial schemes.

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For Johnson schemes,

$$\text{(designs \& constant-weight codes)} \longrightarrow \text{(designs)}.$$

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Definition of δ and δ^*

- $C \subseteq \mathbb{F}_q^D$: a code
- χ_C : the characteristic vector of C

Definition

$$\delta := \min\{i \neq 0 : E_i^* \chi_C \neq 0\} \quad \text{(minimum weight)}$$
$$\delta^* := \min\{i \neq 0 : E_i \chi_C \neq 0\}$$

Remark

If C is linear, then δ^* equals the minimum weight of $C^\perp$.

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