Vertex subsets with minimal width and dual width in Q-polynomial distance-regular graphs

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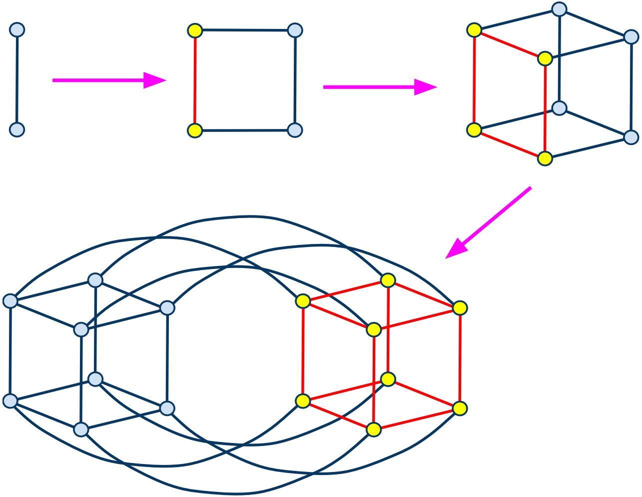
Every face (or facet) of a hypercube is a hypercube...



Goal

- Generalize this situation to Q-polynomial distance-regular graphs.
- Discuss its applications.

Distance-regular graphs

- $\Gamma = (X, R)$: a finite connected simple graph with diameter d
- ∂ : the path-length distance function
- Define $A_0, A_1, \ldots, A_d \in \mathbb{R}^{X \times X}$ by

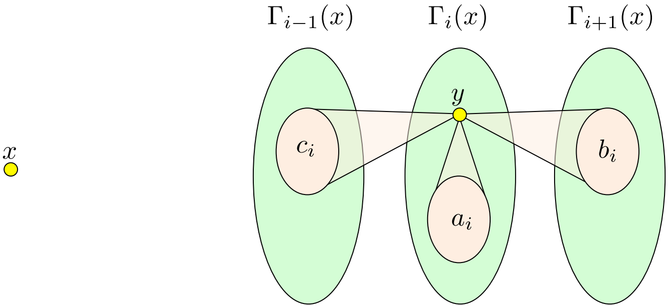
$$(A_i)_{xy} = \begin{cases} 1 & \text{if } \partial(x,y) = i \\ 0 & \text{otherwise} \end{cases}$$

- For $x \in X$, set $\Gamma_i(x) = \{y \in X : \partial(x,y) = i\}$.
- Γ is distance-regular if there are integers a_i, b_i, c_i such that

$$A_1 A_i = b_{i-1} A_{i-1} + a_i A_i + c_{i+1} A_{i+1} \ (0 \leqslant i \leqslant d)$$

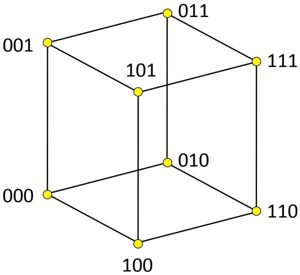
where $A_{-1} = A_{d+1} = 0$.

$A_1 A_i = b_{i-1} A_{i-1} + a_i A_i + c_{i+1} A_{i+1} \quad (0 \leqslant i \leqslant d)$



Example: hypercubes

- $X = \{0,1\}^d$
- $x \sim_R y \iff |\{i : x_i \neq y_i\}| = 1$
- $\Gamma = Q_d = (X, R)$: the hypercube
- Q_3 :

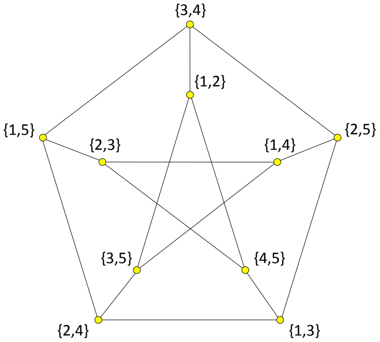


- $Q_d =$ the binary Hamming graph

Example: Johnson graphs

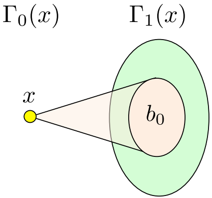
- Ω : a finite set with $|\Omega| = v \geqslant 2d$
- $X = \{x \subseteq \Omega : |x| = d\}$
- $x \sim_R y \iff |x \cap y| = d - 1 \quad (x, y \in \Omega)$
- $\Gamma = J(v, d) = (X, R)$: the Johnson graph

- The complement of $J(5, 2)$ with $\Omega = \{1, 2, 3, 4, 5\}$:



$A_1 A_i = b_{i-1} A_{i-1} + a_i A_i + c_{i+1} A_{i+1} \quad (0 \leqslant i \leqslant d)$

- $\Gamma = (X, R)$: a distance-regular graph with diameter d
- $A_0, A_1, \ldots, A_d$: the distance matrices of Γ
- We set $A := A_1$ (the adjacency matrix of Γ).
- $\theta_0, \theta_1, \ldots, \theta_d$: the distinct eigenvalues of A
- E_i : the orthogonal projection onto the eigenspace of A with eigenvalue θ_i
- $\mathbb{R}[A] = \langle A_0, \ldots, A_d \rangle = \langle E_0, \ldots, E_d \rangle$: the Bose–Mesner algebra of Γ



- Γ is regular with valency $k := b_0$:
- We always set $\theta_0 = k = b_0$.
- $E_0\mathbb{R}^X = \langle \mathbf{1} \rangle$ where $\mathbf{1}$: the all-ones vector
- $E_0 = \frac{1}{|X|}J$ where J : the all-ones matrix in $\mathbb{R}^{X \times X}$

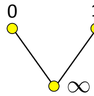
- Recall $A_1 A_i = b_{i-1} A_{i-1} + a_i A_i + c_{i+1} A_{i+1} \quad (0 \leqslant i \leqslant d)$.
- Γ is Q-polynomial with respect to $\{E_i\}_{i=0}^{d}$ if there are scalars a_i^*, b_i^*, c_i^* $(0 \leqslant i \leqslant d)$ such that $b_{i-1}^* c_i^* \neq 0$ $(1 \leqslant i \leqslant d)$ and

$$|X|\ E_1 \circ E_i = b_{i-1}^* E_{i-1} + a_i^* E_i + c_{i+1}^* E_{i+1} \ (0 \leqslant i \leqslant d)$$

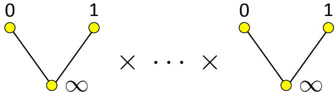
where $E_{-1} = E_{d+1} = 0$ and $\circ$ is the Hadamard product.
- The ordering $\{E_i\}_{i=0}^{d}$ is uniquely determined by E_1.

Hypercubes and binary Hamming matroids

- $\{0, 1, \infty\}$: the "claw semilattice" of order 3 :



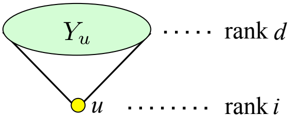
- $(\mathcal{P}, \preccurlyeq)$: the direct product of d claw semilattices:
  - $\mathcal{P} = \{0, 1, \infty\}^d$
  - $u \preccurlyeq v \iff u_i = \infty$ or $u_i = v_i$ $(1 \leqslant i \leqslant d)$



- $H(d, 2) = (\mathcal{P}, \preccurlyeq)$: the binary Hamming matroid
- $\mathrm{rank}(u) = |\{i : u_i \neq \infty\}|$ $(u \in \mathcal{P})$
- $X = \{0, 1\}^d = \mathrm{top}(\mathcal{P})$: the top fiber of $H(d, 2)$

Hypercubes and binary Hamming matroids

- $u \in \mathcal{P}$: rank i
- $\chi_u \in \mathbb{R}^X$: the characteristic vector of $Y_u := \{x \in X : u \preccurlyeq x\}$



Remark

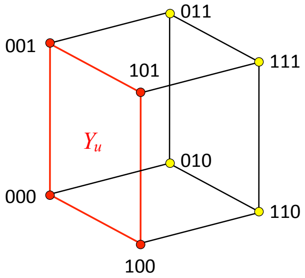
- There is an ordering $E_0, E_1, \ldots, E_d$ such that

$$\sum_{h=0}^{i} E_i \mathbb{R}^X = \langle \chi_u : u \in \mathcal{P},\ \mathrm{rank}(u) = i \rangle \quad (0 \leqslant i \leqslant d).$$

- Moreover, Q_d is Q-polynomial with respect to $\{E_i\}_{i=0}^{d}$.

$Y_u = \{x \in X : u \preccurlyeq x\}$ is a facet of Q_d

- If $d = 3$ and $u = (\infty, 0, \infty)$ then:



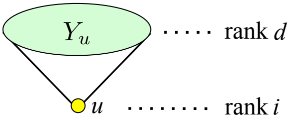
- Every facet of Q_d is of this form.
- The induced subgraph on Y_u is $Q_{d-\mathrm{rank}(u)}$.

Johnson graphs and truncated Boolean algebras

- Recall Ω : a finite set with $|\Omega| = v \geqslant 2d$
- $\mathcal{P} = \{u \subseteq \Omega : |u| \leqslant d\}$
- $u \preccurlyeq v \iff u \subseteq v$
- $B(d, v) = (\mathcal{P}, \preccurlyeq)$: the truncated Boolean algebra
- $\mathrm{rank}(u) = |u| \quad (u \in \mathcal{P})$
- $X = \{x \subseteq \Omega : |x| = d\} = \mathrm{top}(\mathcal{P})$: the top fiber of $B(d, v)$

Johnson graphs and truncated Boolean algebras

- $u \in \mathcal{P}$: rank i
- $\chi_u \in \mathbb{R}^X$: the characteristic vector of $Y_u := \{x \in X : u \preccurlyeq x\}$



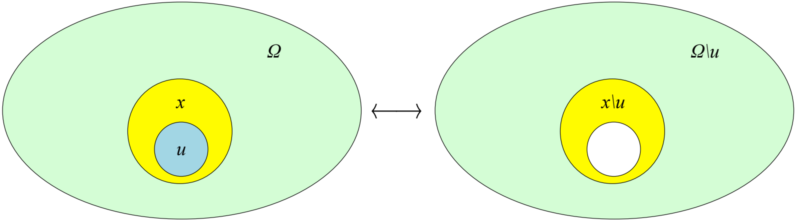
Remark

- There is an ordering $E_0, E_1, \ldots, E_d$ such that

$$\sum_{h=0}^{i} E_i \mathbb{R}^X = \langle \chi_u : u \in \mathcal{P},\ \mathrm{rank}(u) = i \rangle \quad (0 \leqslant i \leqslant d).$$

- Moreover, $J(v,d)$ is Q-polynomial with respect to $\{E_i\}_{i=0}^{d}$.

$Y_u = \{x \in X : u \preccurlyeq x\}$ induces $J(v - \text{rank}(u), d - \text{rank}(u))$



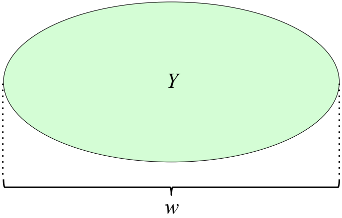
Remark

- $H(d,2)$ and $B(d,v)$ are examples of regular quantum matroids (Terwilliger, 1996).

Width and dual width (Brouwer et al., 2003)

- $\Gamma = (X, R)$: a distance-regular graph with diameter d
- $A_0, A_1, \ldots, A_d$: the distance matrices
- $E_0, E_1, \ldots, E_d$: the primitive idempotents of $\mathbb{R}[A]$
- Suppose Γ is Q-polynomial with respect to $\{E_i\}_{i=0}^{d}$.

- $Y \subseteq X$: a nonempty subset of X
- $\chi \in \mathbb{R}^X$: the characteristic vector of Y
- $w = \max\{i : \chi^{\mathsf{T}} A_i \chi \neq 0\}$: the width of Y
- $w^* = \max\{i : \chi^{\mathsf{T}} E_i \chi \neq 0\}$: the dual width of Y



Theorem (Brouwer–Godsil–Koolen–Martin, 2003)

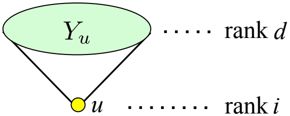
We have $w + w^* \geqslant d$. If equality holds then the induced subgraph Γ_Y on Y is a Q-polynomial distance-regular graph with diameter w provided that it is connected.

Definition

We call Y a descendent of Γ if $w + w^* = d$.

Examples: $\Gamma = Q_d$ or $J(v,d)$

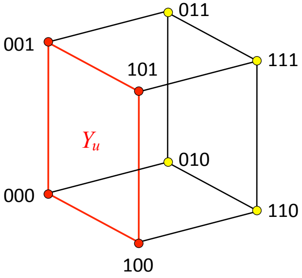
- $u \in \mathcal{P}$: rank i
- $Y_u := \{x \in X : u \preccurlyeq x\}$ satisfies $w = d - i$ and $w^* = i$.



Theorem (Brouwer et al., 2003; T., 2006)

If Γ is associated with a regular quantum matroid, then every descendent of Γ is isomorphic to some Y_u under the full automorphism group of Γ.

Observation



Y_u is convex (geodetically closed).

- We say Γ has classical parameters (d, q, α, β) if

$$b_i = \left(\begin{bmatrix} d \\ 1 \end{bmatrix}_q - \begin{bmatrix} i \\ 1 \end{bmatrix}_q \right) \left(\beta - \alpha \begin{bmatrix} i \\ 1 \end{bmatrix}_q \right), \quad c_i = \begin{bmatrix} i \\ 1 \end{bmatrix}_q \left(1 + \alpha \begin{bmatrix} i-1 \\ 1 \end{bmatrix}_q \right)$$

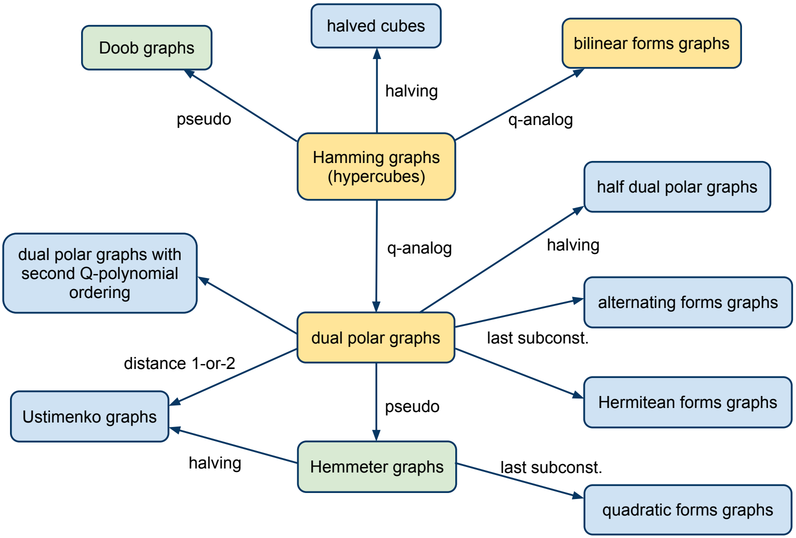
for $0 \leqslant i \leqslant d$, where $\begin{bmatrix} i \\ j \end{bmatrix}_q$ is the q-binomial coefficient.

Example

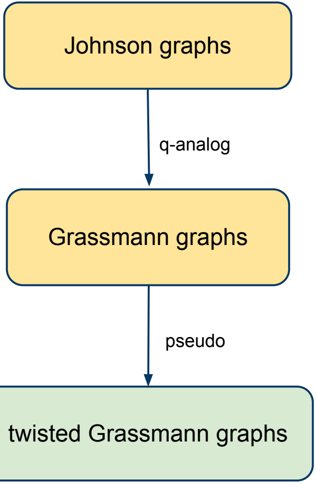
If $\Gamma = Q_d$ then $b_i = d - i$ and $c_i = i$, so Γ has classical parameters $(d, 1, 0, 1)$.

Currently, there are 15 known infinite families of distance-regular graphs with classical parameters and with unbounded diameter.

The families related to Hamming graphs



The families related to Johnson graphs



- $Y \subseteq X$: a descendent of Γ, i.e., $w + w^* = d$
- Γ_Y : the induced subgraph on Y

Theorem (T.)

Suppose $1 < w < d$. Then Y is convex precisely when Γ has classical parameters.

Theorem (T.)

If Γ has classical parameters (d, q, α, β) then Γ_Y has classical parameters (w, q, α, β). The converse also holds, provided $w \geqslant 3$.

Classification of descendents is complete for all 15 families (T.).

The Erdős–Ko–Rado theorem (1961)

- Ω : a finite set with $|\Omega| = v \geqslant 2d$
- $X = \{x \subseteq \Omega : |x| = d\}$

Theorem (Erdős–Ko–Rado, 1961)

Let $v \geqslant (t+1)(d-t+1)$ and let $Y \subseteq X$ be a t-intersecting family, i.e., $|x \cap y| \geqslant t$ for all $x, y \in Y$. Then

$$|Y| \leqslant \binom{v-t}{d-t}.$$

If $v > (t+1)(d-t+1)$ and if $|Y| = \binom{v-t}{d-t}$ then

$$Y = \{x \in X : u \subseteq x\}$$

for some $u \subseteq \Omega$ with $|u| = t$.

A “modern” treatment of the E–K–R theorem

- This is in fact a result about the Johnson graph $J(v,d)$ and the truncated Boolean algebra $B(d,v) = (\mathcal{P}, \preccurlyeq)$.

Theorem (Erdős–Ko–Rado, 1961)

Let $v \geqslant (t+1)(d-t+1)$ and let $Y \subseteq X$ be a t-intersecting family, i.e., $w(Y) \leqslant d-t$. Then

$$|Y| \leqslant \binom{v-t}{d-t}.$$

If $v > (t+1)(d-t+1)$ and if $|Y| = \binom{v-t}{d-t}$ then

$$Y = Y_u$$

for some $u \in \mathcal{P}$ with $rank(u) = t$.

Delsarte's linear programming method

- Define $Q = (Q_{ij})_{0 \leqslant i,j \leqslant d}$ by

$$E_j = \frac{1}{|X|} \sum_{i=0}^{d} Q_{ij} A_i \quad (0 \leqslant j \leqslant d),$$

  or equivalently

$$(E_0, E_1, \ldots, E_d) = \frac{1}{|X|}(A_0, A_1, \ldots, A_d) Q.$$

- Since $E_0 = \frac{1}{|X|} J = \frac{1}{|X|}(A_0 + A_1 + \cdots + A_d)$ we find

$$Q_{00} = Q_{10} = \cdots = Q_{d0} = 1.$$

- $Y \subseteq X$: $w(Y) \leqslant d - t$
- $\chi \in \mathbb{R}^X$: the characteristic vector of Y
- $\boldsymbol{e} = (e_0, e_1, \ldots, e_d)$: the inner distribution of Y :

$$e_i = \frac{1}{|Y|}\chi^{\mathsf{T}} A_i \chi \quad (0 \leqslant i \leqslant d)$$

- Then

(P0) $$(\boldsymbol{e}Q)_0 = e_0 + e_1 + \cdots + e_d = \frac{1}{|Y|}\chi^{\mathsf{T}} J \chi = |Y|,$$

(P1) $$e_0 = 1,$$

(P2) $$e_{d-t+1} = \cdots = e_d = 0,$$

(P3) $$(\boldsymbol{e}Q)_j = \sum_{i=0}^{d} e_i Q_{ij} = \frac{|X|}{|Y|}\chi^{\mathsf{T}} E_j \chi \geqslant 0 \ (1 \leqslant j \leqslant d).$$

$(\boldsymbol{e}Q)_0 = |Y|,\ e_0 = 1,\ e_{d-t+1} = \cdots = e_d = 0,\ (\boldsymbol{e}Q)_j \geqslant 0\ (\forall j)$

- A vector $\boldsymbol{f}$ (unique, if any) satisfying the following conditions gives a feasible solution to the dual problem:

(D1) $$f_0 = 1,$$
(D2) $$f_1 = \cdots = f_t = 0,$$
(D3) $$f_{t+1} > 0, \ldots, f_d > 0,$$
(D4) $$(\boldsymbol{f}Q^{\mathsf{T}})_1 = \cdots = (\boldsymbol{f}Q^{\mathsf{T}})_{d-t} = 0.$$

- By the duality of linear programming, we have

$$|Y| \leqslant (\boldsymbol{f}Q^{\mathsf{T}})_0$$

and equality holds if and only if

$$(\boldsymbol{e}Q)_j f_j = 0\ (1 \leqslant j \leqslant d) \iff (\boldsymbol{e}Q)_{t+1} = \cdots = (\boldsymbol{e}Q)_d = 0$$
$$\iff w^*(Y) \leqslant t.$$

$|Y| \leqslant (fQ^{\mathsf{T}})_0$; $|Y| = (fQ^{\mathsf{T}})_0 \Leftrightarrow w^*(Y) \leqslant t$

- Since $w(Y) \leqslant d - t$ and $w(Y) + w^*(Y) \geqslant d$, we find $|Y| = (fQ^{\mathsf{T}})_0$ if and only if Y is a descendent of $J(v,d)$.
- Under certain conditions, the vector satisfying (D1)–(D4) was constructed in each of the following cases:

Γ	f	$(fQ^{\mathsf{T}})_0$
Johnson $J(v,d)$	Wilson (1984)	$\binom{v-t}{d-t}$
Hamming $H(d,q)$	MDS weight enumerators	q^{d-t}
Grassmann $J_q(v,d)$	Frankl–Wilson (1986)	$\begin{bmatrix} v-t \\ d-t \end{bmatrix}_q$
bilinear forms $\mathrm{Bil}_q(d,e)$	(d,e,t,q)-Singleton systems, Delsarte (1978)	$q^{(d-t)e}$

- Since $J_q(2d+1,d)$ and the twisted Grassmann graph $\tilde{J}_q(2d+1,d)$ have the same Q, we now also get the Erdős–Ko–Rado theorem for $\tilde{J}_q(2d+1,d)$.