A cross-intersection theorem for vector spaces based on semidefinite programming

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The q-Erdős–Ko–Rado theorem

- V : a vector space/$\mathbb{F}_q$ with $\dim V = n$
- $\begin{bmatrix} V \\ k \end{bmatrix} := \{x \leqslant V : \dim x = k\}$
- $\mathfrak{F} \subseteq \begin{bmatrix} V \\ k \end{bmatrix}$: t-intersecting $\overset{\text{def}}{\iff} \dim x \cap y \geqslant t \quad (\forall x, y \in \mathfrak{F})$

Theorem (Hsieh, 1975; T., 2006)

- $\mathfrak{F} \subseteq \begin{bmatrix} V \\ k \end{bmatrix}$: t-intersecting, where $n \geqslant 2k$
- Then

$$|\mathfrak{F}| \leqslant \begin{bmatrix} n-t \\ k-t \end{bmatrix}. \qquad (\#)$$

- Equality holds in (#) $\iff$ One of the following holds:
  1. $\exists z \in \begin{bmatrix} V \\ t \end{bmatrix}$ s.t. $\mathfrak{F} = \left\{x \in \begin{bmatrix} V \\ k \end{bmatrix} : z \subseteq x\right\}$,
  2. $n = 2k$, and $\exists z \in \begin{bmatrix} V \\ 2k-t \end{bmatrix}$ s.t. $\mathfrak{F} = \left\{x \in \begin{bmatrix} V \\ k \end{bmatrix} : x \subseteq z\right\}$.

The q-Erdős–Ko–Rado theorem

Remark

- The bound (#) (for all n, k, q, t) is due to Frankl and Wilson (1986).
- For $t = 1$, the q-EKR theorem was proved independently by Godsil and Newman (2006).

Our goal

- $\mathcal{F} \subseteq \begin{bmatrix} V \\ k \end{bmatrix}$, $\mathcal{G} \subseteq \begin{bmatrix} V \\ \ell \end{bmatrix}$: cross-(1-)intersecting
$\overset{\text{def}}{\Longleftrightarrow} x \cap y \neq 0 \ \ (\forall x \in \mathcal{F}, \forall y \in \mathcal{G})$

Theorem (Suda–T., 2013)

- $\mathcal{F} \subseteq \begin{bmatrix} V \\ k \end{bmatrix}$, $\mathcal{G} \subseteq \begin{bmatrix} V \\ \ell \end{bmatrix}$: cross-intersecting, where $n \geqslant 2k, 2\ell$
- Then

$$|\mathcal{F}|\,|\mathcal{G}| \leqslant \begin{bmatrix} n-1 \\ k-1 \end{bmatrix} \begin{bmatrix} n-1 \\ \ell-1 \end{bmatrix}. \qquad (\flat)$$

- Equality holds in ($\flat$) $\Longleftrightarrow$ One of the following holds:
  1. $\exists z \in \begin{bmatrix} V \\ 1 \end{bmatrix}$ s.t. $\mathcal{F} = \left\{ x \in \begin{bmatrix} V \\ k \end{bmatrix} : z \subseteq x \right\}$, $\mathcal{G} = \left\{ x \in \begin{bmatrix} V \\ \ell \end{bmatrix} : z \subseteq x \right\}$,
  2. $n = 2k = 2\ell$, and $\exists z \in \begin{bmatrix} V \\ 2k-1 \end{bmatrix}$ s.t. $\mathcal{F} = \mathcal{G} = \left\{ x \in \begin{bmatrix} V \\ k \end{bmatrix} : x \subseteq z \right\}$.

Our goal

- This is a q-analogue of a result of Pyber (1986) and Matsumoto and Tokushige (1989), the proof of which uses the Kruskal–Katona theorem.
- Our proof is algebraic in nature and uses the duality of semidefinite programming.

How to prove the q-EKR theorem (Part I)

For simplicity, we assume $t = 1$.

- $\Gamma_k = qK_{n:k}$: the q-Kneser graph
  - $V(\Gamma_k) = \begin{bmatrix} V \\ k \end{bmatrix}$
  - $E(\Gamma_k) = \left\{(x, y) : x, y \in \begin{bmatrix} V \\ k \end{bmatrix},\ x \cap y = 0\right\}$
- $A_k \in \mathbb{R}^{\begin{bmatrix} V \\ k \end{bmatrix} \times \begin{bmatrix} V \\ k \end{bmatrix}}$: the adjacency matrix of Γ_k:

$$(A_k)_{xy} := \begin{cases} 1 & \text{if } x \sim y \\ 0 & \text{otherwise} \end{cases} \quad \left(x, y \in \begin{bmatrix} V \\ k \end{bmatrix}\right)$$

- $I_k \in \mathbb{R}^{\begin{bmatrix} V \\ k \end{bmatrix} \times \begin{bmatrix} V \\ k \end{bmatrix}}$: the identity matrix
- $J_k \in \mathbb{R}^{\begin{bmatrix} V \\ k \end{bmatrix} \times \begin{bmatrix} V \\ k \end{bmatrix}}$: the all 1's matrix
- $Y \bullet Z := \operatorname{trace}(Y^{\mathsf{T}} Z)$ $\left(\forall Y, Z \in \mathbb{R}^{\begin{bmatrix} V \\ k \end{bmatrix} \times \begin{bmatrix} V \\ k \end{bmatrix}}\right)$

How to prove the q-EKR theorem (Part I)

- $\mathcal{F} \subseteq \begin{bmatrix} V \\ k \end{bmatrix}$: a (1-)intersecting family
  $\Longleftrightarrow$ an independent set of Γ_k
- $\varphi \in \mathbb{R}^{\left[V \atop k \right]}$: the (column) characteristic vector of $\mathcal{F}$:

$$\varphi_x = \begin{cases} 1 & \text{if } x \in \mathcal{F} \\ 0 & \text{otherwise} \end{cases} \quad \left(x \in \begin{bmatrix} V \\ k \end{bmatrix}\right)$$

- $X := \frac{1}{||\varphi||^2} \varphi \varphi^{\mathsf{T}} \in \mathbb{R}^{\left[V \atop k \right] \times \left[V \atop k \right]}$: nonnegative & positive semidefinite
- $X \bullet I_k = 1, \; X \bullet J_k = |\mathcal{F}|$
- $X \bullet A_k = \frac{1}{||\varphi||^2} \varphi^{\mathsf{T}} A_k \varphi = 0$

How to prove the q-EKR theorem (Part I)

- Consider the following SDP problem:

$$\vartheta_k = \max_X \, X \bullet J_k$$

subject to

1. $X \bullet I_k = 1$,
2. $X \bullet A_k = 0$,
3. X : nonnegative & positive semidefinite.

- Then $|\mathcal{F}| \leqslant \vartheta_k$.

Remark

$\vartheta_k =$ the strengthening of Lovász's ϑ-function bound due to Schrijver (1979).

How to prove our theorem (Part I)

(Caution: If $k = \ell$ then we view $\begin{bmatrix} V \\ k \end{bmatrix}$ and $\begin{bmatrix} V \\ \ell \end{bmatrix}$ as distinct copies.)

- $\Gamma_{k,\ell}$: a bipartite graph
  - $V(\Gamma_{k,\ell}) = \begin{bmatrix} V \\ k \end{bmatrix} \cup \begin{bmatrix} V \\ \ell \end{bmatrix}$
  - $E(\Gamma_{k,\ell}) = \left\{ (x,y),\ (y,x) : x \in \begin{bmatrix} V \\ k \end{bmatrix},\ y \in \begin{bmatrix} V \\ \ell \end{bmatrix},\ x \cap y = 0 \right\}$
- $A_{k,\ell} \in \mathbb{R}^{\left(\begin{bmatrix} V \\ k \end{bmatrix} \cup \begin{bmatrix} V \\ \ell \end{bmatrix}\right) \times \left(\begin{bmatrix} V \\ k \end{bmatrix} \cup \begin{bmatrix} V \\ \ell \end{bmatrix}\right)}$: the adjacency matrix of $\Gamma_{k,\ell}$:

$$A_{k,\ell} = \begin{pmatrix} 0_k & * \\ * & 0_\ell \end{pmatrix}$$

- $J_{k,\ell} \in \mathbb{R}^{\left(\begin{bmatrix} V \\ k \end{bmatrix} \cup \begin{bmatrix} V \\ \ell \end{bmatrix}\right) \times \left(\begin{bmatrix} V \\ k \end{bmatrix} \cup \begin{bmatrix} V \\ \ell \end{bmatrix}\right)}$: the adjacency matrix of the complete bipartite graph with bipartition $\begin{bmatrix} V \\ k \end{bmatrix} \cup \begin{bmatrix} V \\ \ell \end{bmatrix}$:

$$J_{k,\ell} = \begin{pmatrix} 0_k & J \\ J & 0_\ell \end{pmatrix}$$

How to prove our theorem (Part I)

- $\mathcal{F} \subseteq \begin{bmatrix} V \\ k \end{bmatrix}$, $\mathcal{G} \subseteq \begin{bmatrix} V \\ \ell \end{bmatrix}$: cross-intersecting families
  $\iff \mathcal{F} \cup \mathcal{G}$: an independent set of $\Gamma_{k,\ell}$
- $\varphi \in \mathbb{R}^{\begin{bmatrix} V \\ k \end{bmatrix}}$: the characteristic vector of $\mathcal{F}$
- $\psi \in \mathbb{R}^{\begin{bmatrix} V \\ \ell \end{bmatrix}}$: the characteristic vector of $\mathcal{G}$
- $X := \begin{pmatrix} \frac{1}{\|\varphi\|^2}\varphi\varphi^{\mathsf{T}} & \frac{1}{\|\varphi\|\,\|\psi\|}\varphi\psi^{\mathsf{T}} \\ \frac{1}{\|\varphi\|\,\|\psi\|}\psi\varphi^{\mathsf{T}} & \frac{1}{\|\psi\|^2}\psi\psi^{\mathsf{T}} \end{pmatrix} \in \mathbb{R}^{\left(\begin{bmatrix} V \\ k \end{bmatrix} \cup \begin{bmatrix} V \\ \ell \end{bmatrix}\right) \times \left(\begin{bmatrix} V \\ k \end{bmatrix} \cup \begin{bmatrix} V \\ \ell \end{bmatrix}\right)}$:

  nonnegative & positive semidefinite
- $X \bullet I_k = X \bullet I_\ell = 1,\ \frac{1}{2} X \bullet J_{k,\ell} = \sqrt{|\mathcal{F}|\,|\mathcal{G}|}$
- $X \bullet A_{k,\ell} = 0$

How to prove our theorem (Part I)

- Consider the following SDP problem:

$$\vartheta_{k,\ell} = \frac{1}{2} \max_{X} X \bullet J_{k,\ell}$$

subject to

1. $X \bullet I_k = X \bullet I_\ell = 1$,
2. $X \bullet A_{k,\ell} = 0$,
3. X : nonnegative & positive semidefinite.

- Then $|\mathcal{F}|\,|\mathcal{G}| \leqslant (\vartheta_{k,\ell})^2$.

$\vartheta_{k,\ell}$ is a "bipartite variant" of Lovász's ϑ-function bound !!

How to prove the q-EKR theorem (Part II)

- We prove the EKR bound $|\mathcal{F}| \leqslant \left[{n-1 \atop k-1}\right]$ by constructing an optimal feasible solution to the dual program of ϑ_k.
- To this end, we notice that

$$I_k, J_k, A_k \in \mathcal{A}$$

  where $\mathcal{A} =$ the Bose–Mesner algebra of the Grassmann graph $J_q(n,k)$ ($=$ the commutant of $GL(V)$ acting on $\left[{V \atop k}\right]$)
- By projecting the variable X to $\mathcal{A}$, we may assume

$$X \in \mathcal{A}.$$

- Since $\mathcal{A}$ is a commutative matrix $*$-algebra, it is simultaneously diagonalized.

  $\Rightarrow \vartheta_k$ turns to an LP !!

How to prove the q-EKR theorem (Part II)

- The dual program is given by

$$\vartheta_k = \min_{\alpha,\gamma,Z} \alpha$$

subject to

1. $\alpha I_k - \gamma A_k - J_k - Z$: positive semidefinite,
2. Z : nonnegative.

- After the reduction, it can be shown that there is a unique optimal feasible solution to the dual program:

$$\alpha = \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}, \quad \gamma = -q^{-k^2+k} \cdot \frac{\begin{bmatrix} n-1 \\ k-1 \end{bmatrix}}{\begin{bmatrix} n-k-1 \\ k-1 \end{bmatrix}}, \quad Z = 0.$$

(More on this later.)

How to prove our theorem (Part II)

- We prove the bound $|\mathcal{F}|\,|\mathcal{G}| \leqslant \begin{bmatrix} n-1 \\ k-1 \end{bmatrix} \begin{bmatrix} n-1 \\ \ell-1 \end{bmatrix}$ by constructing an optimal feasible solution to the dual program of $\vartheta_{k,\ell}$.
- To this end, we notice that

$$I_k, I_\ell, J_{k,\ell}, A_{k,\ell} \in \mathcal{C}$$

where $\mathcal{C} =$ the coherent algebra of the commutant of $GL(V)$ acting on $\begin{bmatrix} V \\ k \end{bmatrix} \cup \begin{bmatrix} V \\ \ell \end{bmatrix}$

- By projecting the variable X to $\mathcal{C}$, we may assume

$$X \in \mathcal{C}.$$

- $\mathcal{C}$ is a matrix $*$-algebra, and it is simultaneously block-diagonalized into (at most) 2×2 matrices.
$\Rightarrow \vartheta_{k,\ell}$ turns to a drastically smaller SDP !!

How to prove our theorem (Part II)

- The dual program is given by

$$\vartheta_{k,\ell} = \min_{\alpha,\beta,\gamma,Z} \alpha + \beta$$

subject to

1. $\alpha I_k + \beta I_\ell - \gamma A_{k,\ell} - \frac{1}{2} J_{k,\ell} - Z$: positive semidefinite,
2. Z : nonnegative.

- After the reduction, it can be shown that there is a unique one-parameter family of optimal feasible solutions to the dual program:

$$\alpha = \beta = \frac{1}{2} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}^{\frac{1}{2}} \begin{bmatrix} n-1 \\ \ell-1 \end{bmatrix}^{\frac{1}{2}}, \quad \gamma = b(\lambda), \quad Z = a(\lambda) A_k + \lambda A_\ell,$$

where we are assuming $k \geqslant \ell$, and ... (!)

How to prove our theorem (Part II)

... the functions $a(\lambda)$ and $b(\lambda)$ are given by

$$q^{k^2}(q^k-1)\begin{bmatrix} n-k \\ k \end{bmatrix} \cdot a(\lambda) = \frac{1}{2}q^{\ell}(q^{k-\ell}-1)\begin{bmatrix} n-1 \\ k-1 \end{bmatrix}^{\frac{1}{2}}\begin{bmatrix} n-1 \\ \ell-1 \end{bmatrix}^{\frac{1}{2}} + q^{\ell^2}(q^{\ell}-1)\begin{bmatrix} n-\ell \\ \ell \end{bmatrix}\lambda,$$

$$q^{k\ell}\begin{bmatrix} n-k \\ \ell \end{bmatrix} \cdot b(\lambda) = -\frac{1}{2}q^{\ell}\begin{bmatrix} n-1 \\ \ell \end{bmatrix} - q^{\ell^2}\begin{bmatrix} n-\ell \\ \ell \end{bmatrix}\begin{bmatrix} n-1 \\ \ell-1 \end{bmatrix}^{\frac{1}{2}}\begin{bmatrix} n-1 \\ k-1 \end{bmatrix}^{-\frac{1}{2}}\lambda$$

for sufficiently small $\lambda \geqslant 0$.

Compare this with the unique solution of ϑ_k !!

End of the proofs

- In both of the theorems, the optimal feasible solutions to the dual programs, together with the duality of LP / SDP, provide enough information about the characteristic vectors of optimal (cross-)intersecting families.
- By “enough information” I do NOT mean that the characterization of optimal families is easy.

  [Recall that the q-EKR theorem was proved in full generality only in 2006.]
- Indeed, the discussions in this part are very typical in Delsarte theory.

Delsarte theory (1973)

- studies in a unified manner various combinatorial objects (e.g., codes, designs) whose underlying spaces have "strong" symmetry / regularity.
- bounds the value of a numerical parameter (e.g., size, index) of such objects.
- shows that optimal (or nearly optimal) objects satisfy certain additional regularity.
- then in some cases classifies the optimal (or nearly optimal) objects.

Example

- Objects : intersecting families $\mathcal{F}$
- Underlying space : $\begin{bmatrix} V \\ k \end{bmatrix} \curvearrowleft GL(V)$

“Classical” Delsarte theory

- Commutative matrix $*$-algebras (Bose–Mesner algebras)
- Linear programming

Example

- LP bound on the size of a code (Delsarte, 1972)
  - sphere-packing bound
  - Singleton bound
  - Plotkin bound
  - McEliece–Rodemich–Rumsey–Welch bound
- Lloyd’s theorem (Delsarte, 1973)
- Erdős–Ko–Rado theorem (Wilson, 1984)

“Quantum” Delsarte theory (still in its infancy)

- Noncommutative matrix $*$-algebras (e.g., coherent algebras, Terwilliger algebras)
- Semidefinite programming

Example

- SDP bound on the size of a code (Schrijver, 2005; Gijswijt–Schrijver–T., 2006)
- Today’s theorem

EKR theorems for distance-regular graphs

- $J_q(n,k)$: the Grassmann graph:
  - $V(J_q(n,k)) = \begin{bmatrix} V \\ k \end{bmatrix}$
  - $E(J_q(n,k)) = \left\{(x,y) : x,y \in \begin{bmatrix} V \\ k \end{bmatrix}, \dim x \cap y = k-1\right\}$
- It follows that

  $$x, y : \text{at distance } i \iff \dim x \cap y = k - i$$
- Hence $\Gamma_k = qK_{n:k}$: the distance-k graph of $J_q(n,k)$

$J_q(n,k)$ is an example of a Q-polynomial distance-regular graph:

"very strong regularity" + "very nice structure of the eigenspaces"

EKR theorems for distance-regular graphs

- Every Q-polynomial DRG Γ (with diameter k) is associated with the parameter array:

$$\left(\{\theta_i\}_{i=0}^{k}; \{\theta_i^*\}_{i=0}^{k}; \{\varphi_i\}_{i=1}^{k}; \{\phi_i\}_{i=1}^{k}\right).$$

[The θ_i are the distinct eigenvalues of Γ.]

Example

For $J_q(n,k)$, the parameter array is of the dual q-Hahn type:

- $\theta_i = \theta_0 + h(1-q^i)(1-sq^{i+1})q^{-i} \quad (0 \leqslant i \leqslant k)$
- $\theta_i^* = \theta_0^* + h^*(1-q^i)q^{-i} \quad (0 \leqslant i \leqslant k)$
- $\varphi_i = hh^*q^{1-2i}(1-q^i)(1-q^{i-k-1})(1-rq^i) \quad (1 \leqslant i \leqslant k)$
- $\phi_i = hh^*q^{k+2-2i}(1-q^i)(1-q^{i-k-1})(s-rq^{i-k-1}) \quad (1 \leqslant i \leqslant k)$

- The SDP problem ϑ_k can be defined for t-intersecting families ($t \geqslant 2$) as well, and for any Q-polynomial DRGs.
- After the reduction to LP:

  "X : positive semidefinite"

  $\rightarrow$ $k+1$ nonnegativity constraints (indexed $0, 1, \ldots, k$)

- $f_j =$ the j^{th} component of the unique optimal feasible solution to the dual LP ($0 \leqslant j \leqslant k$)

EKR theorems for distance-regular graphs

Theorem (T., 2012)

$f_0 = 1,\ f_1 = \cdots = f_t = 0$, *and*

$$f_j = \frac{\eta_{k-t}(\theta_0)}{\eta_k(\theta_0)\eta^*_t(\theta^*_0)} \frac{\phi_{k-j+1}\ldots\phi_k}{\varphi_2\ldots\varphi_j(\theta_j-\theta_0)} \left(\sum_{\ell=t+1}^{j} \frac{\tau_\ell(\theta_j)\eta^*_{\ell-1}(\theta^*_0)\vartheta_\ell}{\phi_{k-\ell+1}\ldots\phi_{k-t}}\right)$$

for $t+1 \leqslant j \leqslant k$, *where*

$$\tau_i(z) = (z-\theta_0)\ldots(z-\theta_{i-1}), \quad \eta_i(z) = (z-\theta_d)\ldots(z-\theta_{d-i+1}),$$

$$\tau^*_i(z) = (z-\theta^*_0)\ldots(z-\theta^*_{i-1}), \quad \eta^*_i(z) = (z-\theta^*_d)\ldots(z-\theta^*_{d-i+1}),$$

$$\vartheta_i = \sum_{h=0}^{i-1} \frac{\theta_h-\theta_{d-h}}{\theta_0-\theta_d}.$$

Remark

- The f_j $(t+1 \leqslant j \leqslant k)$ are expressed as ${}_4\phi_3$ basic hypergeometric series (including their special / limiting cases).
- Using this result, the EKR theorem can be proved in a unified manner for several families of Q-polynomial DRGs (T., 2012), e.g.,
  - Johnson graphs (Wilson, 1984) $\leftrightarrow$ original EKR
  - Grassmann graphs (Hsieh, 1975; T., 2006) $\leftrightarrow$ q-EKR
  - Hamming graphs (Moon, 1982) $\leftrightarrow$ integer sequences
  - bilinear forms graphs (Huang, 1987; T., 2006)
  - twisted Grassmann graphs (T., 2012)

Summary & future work

	EKR $(t = 1)$	cross-intersection
constraints	only 1×1 matrices (i.e., LP)	involve 2×2 matrices
optimal solutions to dual program	unique	1-parameter family

Summary & future work

	EKR $(t \geqslant 2)$	cross t-intersection
constraints	only 1×1 matrices (i.e., LP)	involve 2×2 matrices
optimal solutions to dual program	unique	t-parameter family

The description of the above t-parameter family involves 3 parameter arrays, and it is too complicated to be stated here.