On tight relative t-designs in hypercubes

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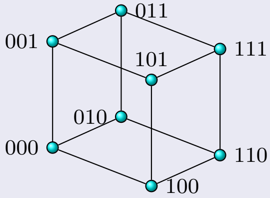
Relative t-designs in the n-cube $\mathcal{Q}_n$

- $[n] := \{1, 2, \ldots, n\} \quad (n \in \mathbb{N})$
- $\mathcal{Q}_n := 2^{[n]} = \{x : x \subseteq [n]\}$: the n-cube

  A structure will be given later

- $\binom{[n]}{k} = \{x \in 2^{[n]} : |x| = k\}$
- $\emptyset \neq Y \subset 2^{[n]}$
- $\omega : Y \to \mathbb{R}_{>0}$

- $\{0,1\}^n \overset{1:1}{\longleftrightarrow} 2^{[n]}$
  - $1010\cdots0 \longleftrightarrow \{1,3\}$
  - $1110\cdots0 \longleftrightarrow \{1,2,3\}$



Definition (Delsarte (1977))

- (Y, ω) : a relative t-design ← "weighted" regular t-wise balanced design

  $\overset{\text{def}}{\Longleftrightarrow} \exists \lambda_1, \lambda_2, \ldots, \lambda_t \in \mathbb{R}_{>0}$ s.t. for $i = 1, 2, \ldots, t$,

$$\sum_{z \subset y \in Y} \omega(y) = \lambda_i \qquad \left(\forall z \in \binom{[n]}{i}\right)$$

Delsarte's design theory

$|L| = 1$ (t-designs)
Delsarte (1973)

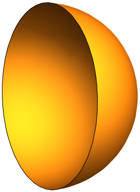
$\longrightarrow$

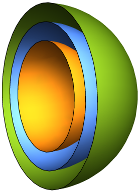
$|L| \geqslant 2$ (relative t-designs)
Delsarte (1977)

spherical t-designs
Delsarte–Goethals–Seidel (1977)

$\longrightarrow$

Euclidean t-designs
Neumaier–Seidel (1988)





Tight relative t-designs

Theorem (Bannai–Bannai (2012); Xiang (2012))

- (Y, ω) *: a relative* $2e$*-design*
- $L := \{\ell : Y \cap \binom{[n]}{\ell} \neq \emptyset\}$ *[+ additional assumptions on* e *and* L *]*
- *Then*

$$|Y| \geqslant \binom{n}{e} + \binom{n}{e-1} + \cdots + \binom{n}{e-|L|+1}.$$

Fisher-type inequality

Definition

- (Y, ω) : tight $\overset{\text{def}}{\Longleftrightarrow}$ $|Y| = \binom{n}{e} + \binom{n}{e-1} + \cdots + \binom{n}{e-|L|+1}$

The tight case with $|L| = 1$

Remark (possibly incorrect and/or incomplete)

- Ray-Chaudhuri and Wilson (1975): $|Y| \geqslant \binom{n}{e}$

Tight implies a Hahn polynomial has only integral zeros.

- Ito (1975), Bremner (1979): only 2 examples for $e = 2$
- Peterson (1977): none for $e = 3$
- Bannai (1977): finite for $e \geqslant 5$
- Dukes–Short-Gershman (2013): none for $e = 5, 6, 7, 8, 9$
- Xiang (unpublished?): none for $e = 4$

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Main result I (the tight case with $|L| = 2$)

Theorem

- $L = \{\ell, m\}$ *where* $e \leqslant \ell < m \leqslant n - e$
- (Y, ω) *: tight relative* $2e$*-design with* $Y \subseteq \binom{[n]}{\ell} \sqcup \binom{[n]}{m}$
- *Then the polynomial*

$$\psi_e^{\ell,m}(\xi) := {}_3F_2\left(\begin{matrix} -\xi, -e, e-n-1 \\ m-n, -\ell \end{matrix} \,\middle|\, 1\right) \in \mathbb{R}[\xi]$$

with degree e *has only integral zeros.*

a Hahn polynomial

Main result I (the tight case with $|L| = 2$)

Example

- Bannai–Bannai–Zhu (2016+) found four tight relative 4-designs for $n = 22$:

n	ℓ	m	ξ
22	6	7	$3, 5$
22	6	15	$1, 3$
22	7	16	$1, 3$
22	15	16	$3, 5$

- The zeros ξ are integers!!

Main result I (the tight case with $|L| = 2$)

Example

- The existence of tight relative 4-designs with the following feasible parameters were left open:

n	ℓ	m	ξ
37	9	16	$\frac{1}{14}(71 \pm \sqrt{337})$
37	9	21	$\frac{1}{14}(55 \pm \sqrt{337})$
37	16	28	$\frac{1}{14}(55 \pm \sqrt{337})$
37	21	28	$\frac{1}{14}(71 \pm \sqrt{337})$
41	15	16	$\frac{1}{26}(237 \pm \sqrt{1569})$
41	15	25	$\frac{1}{26}(153 \pm \sqrt{1569})$
41	16	26	$\frac{1}{26}(153 \pm \sqrt{1569})$
41	25	26	$\frac{1}{26}(237 \pm \sqrt{1569})$

- The zeros ξ are irrational, thus proving the non-existence!!

An algebraic approach

- X : a finite set
- $\mathbb{C}^{X \times X}$: the $\mathbb{C}$-algebra of matrices indexed by X
- I : the identity matrix
- J : the all 1's matrix

entrywise (Hadamard) product

- $\boldsymbol{A} \subseteq \mathbb{C}^{X \times X}$: a subalgebra consisting of symmetric matrices
- $\boldsymbol{A}$: a Bose–Mesner algebra $\overset{\text{def}}{\iff}$
  1. $I, J \in \boldsymbol{A}$
  2. $\boldsymbol{A}$: closed under $\circ$

always commutative

An algebraic approach

- $\boldsymbol{H} \subseteq \mathbb{C}^{X \times X}$: a subalgebra
- $\boldsymbol{H}$: a coherent algebra $\overset{\text{def}}{\Longleftrightarrow}$
  1. $I, J \in \boldsymbol{H}$
  2. $\boldsymbol{H}$: closed under $\circ$
  3. $\boldsymbol{H}$: closed under transposition
- I is a sum of diagonal 01-matrices in $\boldsymbol{H}$:

$$I = \left[\begin{array}{c|c|c} \begin{smallmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{smallmatrix} & & \\ \hline & & \\ \hline & & \end{array}\right] + \cdots + \left[\begin{array}{c|c|c} & & \\ \hline & & \\ \hline & & \begin{smallmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{smallmatrix} \end{array}\right]$$

$\Longrightarrow \; X = X_0 \sqcup \cdots \sqcup X_n$: the fiber decomposition

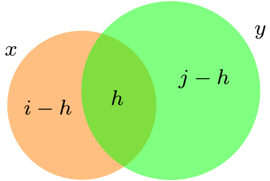
Remark

- Every Bose–Mesner algebra has only one fiber.

The Terwilliger algebra of $\mathcal{Q}_n$

- For $i, j, h = 0, 1, \ldots, n$, define $A_{i,j}^h \in \mathbb{C}^{2^{[n]} \times 2^{[n]}}$ by

$$A_{i,j}^h(x, y) = \begin{cases} 1 & \text{if } |x| = i, \ |y| = j, \ |x \cap y| = h, \\ 0 & \text{otherwise.} \end{cases}$$



Fact

- $\boldsymbol{T} := \mathrm{span}\{A_{i,j}^h\}_{i,j,h=0}^n$: a coherent algebra
- $2^{[n]} = \binom{[n]}{0} \sqcup \binom{[n]}{1} \sqcup \cdots \sqcup \binom{[n]}{n}$: the fiber decomposition
- $\boldsymbol{T}$: the Terwilliger algebra of $\mathcal{Q}_n$ $=$ the commutant of $\mathfrak{S}_n \curvearrowright 2^{[n]}$

non-commutative

The tight case with $|L| = 1$, revisited

Remark

- $L = \{\ell\}$
- $\boldsymbol{A}_\ell := \mathrm{span}\{A^h_{\ell,\ell}\}_{h=0}^n \; (\subseteq \mathbb{C}^{\binom{[n]}{\ell} \times \binom{[n]}{\ell}})$: a Bose–Mesner algebra
- $\boldsymbol{A}_\ell$: the Bose–Mesner algebra of the Johnson scheme on $\binom{[n]}{\ell}$
- (Y, ω) : tight relative $2e$-design with $Y \subseteq \binom{[n]}{\ell}$
- Delsarte (1973) showed that

$$\boldsymbol{A}_\ell|_{Y\times Y} := \{B|_{Y\times Y} : B \in \boldsymbol{A}_\ell\}$$

principal submatrix

is a Bose–Mesner algebra.

commutative

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Main result II (the tight case with $|L| = 2$)

Theorem

- $L = \{\ell, m\}$ *where* $e \leqslant \ell < m \leqslant n - e$
- (Y, ω) *: tight relative $2e$-design with* $Y \subseteq \binom{[n]}{\ell} \sqcup \binom{[n]}{m}$
- *Then*

$$\boldsymbol{T}|_{Y\times Y} := \left\{ B|_{Y\times Y} : B \in \boldsymbol{T} \right\}$$

is a coherent algebra.

non-commutative

$\lvert L\rvert = 1$ (t-designs)	→	$\lvert L\rvert = 2$ (relative t-designs)
Bose–Mesner algebra (commutative)		Terwilliger algebra (non-commutative)

Frequently asked questions

Q : What about the case $|L| \geqslant 3$?

- This seems difficult. (I do not expect that similar results hold.)
- The case $|L| = 2$ is particularly promising:
  - similar results for tight Euclidean designs (Bannai–Bannai (2010))
  - active research on cross-intersecting families (including an SDP extension of Delsarte's LP bound)

Q : What about other Q-polynomial association schemes?

- In general we obtain only weaker results. What is special about $\mathcal{Q}_n$ is that each fiber induces again a Q-polynomial association scheme (a Johnson scheme).