距離正則グラフの Terwilliger 代数の拡張について

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第 63 回代数学シンポジウム

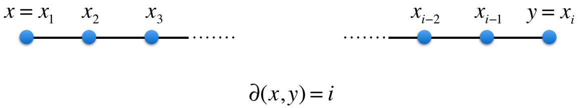
vertex set
edge set

- $\Gamma = (X, R)$: a Q-polynomial distance-regular graph
  - e.g., $(\mathfrak{S}_2 \wr \mathfrak{S}_n)/\mathfrak{S}_n$, $\mathfrak{S}_{n+m}/(\mathfrak{S}_n \times \mathfrak{S}_m)$
  - induces polynomials in the Askey scheme [Leonard (1982)] Go to 29
  - ultimate goal: to classify all such graphs [Bannai–Ito (1984)]
- $X \ni x \longrightarrow \boldsymbol{T} = \boldsymbol{T}(x)$: the Terwilliger algebra $/\mathbb{C}$
  - non-commutative
  - finite-dimensional, semisimple
  - generators A, A^*

dual adjacency matrix
adjacency matrix
tensor products of evaluation modules

- W : an irreducible $\boldsymbol{T}$-module $\longrightarrow$ $A|_W, A^*|_W$: a tridiagonal pair
  - classified when q not a root of unity [Ito–Terwilliger (2010)]
  - constructed in this case from $U_q(\widehat{\mathfrak{sl}}_2)$-modules
  - parametrized in the general case [Ito–Nomura–Terwilliger (2011)]

- $\Gamma = (X, R)$: a finite connected simple graph
  - X : the vertex set
  - R : the edge set ($=$ a set of 2-element subsets of X)
- ∂ : the path-length distance on X



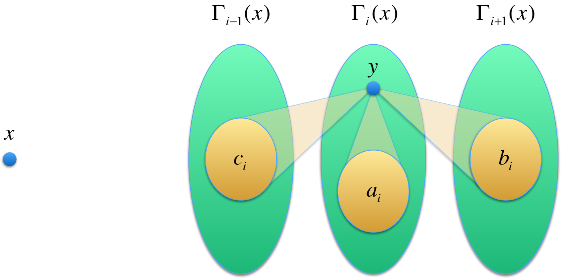
- $D := \max\{\partial(x, y) : x, y \in X\}$: the diameter of Γ
- $\Gamma_i(x) := \{y \in X : \partial(x, y) = i\}$: the i^{th} subconstituent w.r.t. x
- $\Gamma(x) := \Gamma_1(x)$

- Γ : distance-regular

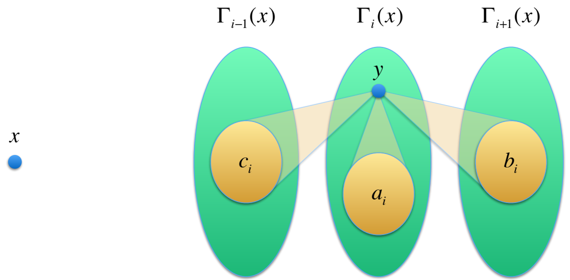
$\overset{\text{def}}{\iff} \exists a_i, b_i, c_i \ (0 \leqslant i \leqslant D)$ s.t. $\forall x, y \in X$:

  - $|\Gamma_{i-1}(x) \cap \Gamma(y)| = c_i$
  - $|\Gamma_i(x) \cap \Gamma(y)| = a_i$
  - $|\Gamma_{i+1}(x) \cap \Gamma(y)| = b_i$

where $\partial(x, y) = i$.



For the rest of the talk, we assume that Γ is distance-regular!!

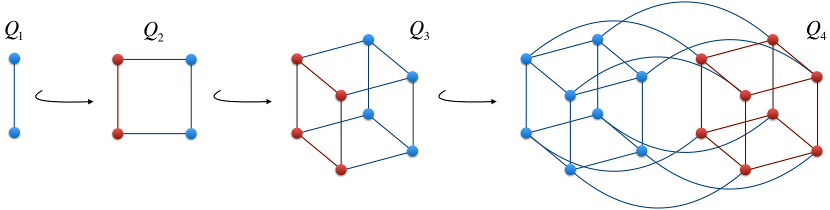


- Γ : regular with valency $k = b_0 = |\Gamma(x)|$
- $\iota(\Gamma) := \{b_0, b_1, \ldots, b_{D-1}; c_1, c_2, \ldots, c_D\}$: the intersection array of Γ

  [Note: $a_i + b_i + c_i = k$]

Example (the D-cube $\mathcal{Q}_D$)

- $X = \{0, 1\}^D$
- $x = (x_1, \ldots, x_D) \sim y = (y_1, \ldots, y_D) \overset{\text{def}}{\Longleftrightarrow} |\{i : x_i \neq y_i\}| = 1$



Remark

- $\iota(\mathcal{Q}_D) = \{D, \ldots, 2, 1; 1, 2, \ldots, D\}$
- $X = (\mathfrak{S}_2 \wr \mathfrak{S}_D)/\mathfrak{S}_D$

- $\mathbb{C}^{X\times X}$: the set of square matrices over $\mathbb{C}$ index by X
- The i^{th} distance matrix $A_i \in \mathbb{C}^{X\times X}$ is

$$(A_i)_{x,y} = \begin{cases} 1 & \text{if } \partial(x,y) = i, \\ 0 & \text{otherwise,} \end{cases} \qquad (x, y \in X).$$

[Note: $A_0 = I$]

- $A := A_1$: the adjacency matrix of Γ
- $A_0, A_1, \ldots, A_D$ satisfy the three-term recurrence

$$A \cdot A_i = b_{i-1}A_{i-1} + a_i A_i + c_{i+1}A_{i+1} \ \ (0 \leqslant i \leqslant D)$$

where $A_{-1} = A_{D+1} := 0$.

- Recall

$$A \cdot A_i = b_{i-1}A_{i-1} + a_iA_i + c_{i+1}A_{i+1} \ \ (0 \leqslant i \leqslant D).$$

- $\boldsymbol{A} := \mathbb{C}[A] \subset \mathbb{C}^{X \times X}$: the adjacency algebra of Γ
  - $\boldsymbol{A} = \langle A_0, A_1, \ldots, A_D \rangle$
  - A has $D+1$ distinct eigenvalues $\theta_0, \theta_1, \ldots, \theta_D \in \mathbb{R}$.
  - $\iota(\Gamma)$ determines $\boldsymbol{A}$ algebraically.

- Recall
  - $\theta_0, \theta_1, \ldots, \theta_D \in \mathbb{R}$: the distinct eigenvalues of A
  - Γ : regular with valency $k = b_0 = |\Gamma(x)|$
- Always set $\theta_0 = k$.
- $E_\ell \in \mathbb{C}^{X \times X}$: the orthogonal projection onto the eigenspace of θ_ℓ

  [Note: $E_0 = \frac{1}{|X|} J$ (J : the all-ones matrix)]
- $\boldsymbol{A} = \mathbb{C}[A] = \langle A_0, A_1, \ldots, A_D \rangle = \langle E_0, E_1, \ldots, E_D \rangle$
- $E_0, E_1, \ldots, E_D$: the primitive idempotents of $\boldsymbol{A}$

- Recall

$$A \cdot A_i = b_{i-1}A_{i-1} + a_iA_i + c_{i+1}A_{i+1} \ \ (0 \leqslant i \leqslant D).$$

- Γ : Q-polynomial w.r.t. $\{E_\ell\}_{\ell=0}^D$ (or $\{\theta_\ell\}_{\ell=0}^D$)

$\overset{\text{def}}{\Longleftrightarrow} \exists a_\ell^*, b_\ell^*, c_\ell^*$ $(0 \leqslant \ell \leqslant D)$ s.t. $b_{\ell-1}^*c_\ell^* \neq 0 \ \ (1 \leqslant \ell \leqslant D)$ and

$$|X|\, E_1 \circ E_\ell = b_{\ell-1}^*E_{\ell-1} + a_\ell^*E_\ell + c_{\ell+1}^*E_{\ell+1} \ \ (0 \leqslant \ell \leqslant D)$$

where $E_{-1} = E_{D+1} := 0$ and $\circ$ is the entrywise product.

- $A_0, A_1, \ldots, A_D$: the primitive idempotents of $\boldsymbol{A}$ w.r.t. $\circ$

Example

- $\Gamma = \mathcal{Q}_D$ is Q-polynomial w.r.t. $\theta_0 > \theta_1 > \cdots > \theta_D$.

We will always assume that Γ is Q-polynomial as above!!

- Recall

$$A \cdot A_i = b_{i-1}A_{i-1} + a_i A_i + c_{i+1}A_{i+1} \quad (0 \leqslant i \leqslant D),$$
$$|X|\, E_1 \circ E_\ell = b^*_{\ell-1}E_{\ell-1} + a^*_\ell E_\ell + c^*_{\ell+1}E_{\ell+1} \quad (0 \leqslant \ell \leqslant D).$$

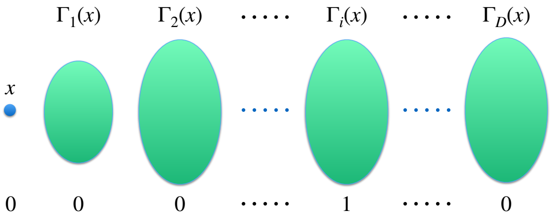
- These three-term recurrences define two systems of orthogonal polynomials $\{f_i\}_{i=0}^D$ and $\{f^*_\ell\}_{\ell=0}^D$.

Theorem (Leonard (1982); Bannai–Ito (1984))

- *The f_i and the f^*_ℓ belong to the Askey scheme.*

- Fix a base vertex $x \in X$.
- The i^{th} dual idempotent $E_i^* = E_i^*(x) \in \mathbb{C}^{X \times X}$ is

$$(E_i^*)_{y,z} = \begin{cases} 1 & \text{if } \ y = z \in \Gamma_i(x), \\ 0 & \text{otherwise}, \end{cases} \qquad (y, z \in X).$$



- $\boldsymbol{T} = \boldsymbol{T}(x) = \mathbb{C}[A, E_0^*, \ldots, E_D^*]$: the Terwilliger algebra of Γ w.r.t. x

- Recall

$$\boldsymbol{T} = \boldsymbol{T}(x) = \mathbb{C}[A, E_0^*, \ldots, E_D^*].$$

- The dual adjacency matrix $A^* = A^*(x) \in \mathbb{C}^{X \times X}$ is

$$(A^*)_{y,z} = \begin{cases} |X|(E_1)_{x,y} & \text{if } y = z, \\ 0 & \text{otherwise,} \end{cases} \qquad (y, z \in X).$$

Lemma

- $\boldsymbol{T} = \mathbb{C}[A, A^*]$.

Proof.

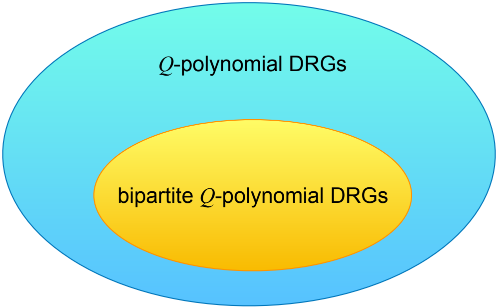
- Recall $|X|\, E_1 \circ E_\ell = b^*_{\ell-1} E_{\ell-1} + a^*_\ell E_\ell + c^*_{\ell+1} E_{\ell+1} \quad (0 \leqslant \ell \leqslant D)$. □

Corollary

- A, A^* *act on irreducible* $\boldsymbol{T}$*-modules as tridiagonal pairs.* ▸ Go to 33

- The representation theory of $\boldsymbol{T}$ has been developed by Terwilliger, Ito, and others;
- ... and has been used, e.g., in
  - the classification of $\iota(\Gamma)$ when Γ is bipartite (Caughman (2004) for $D \geqslant 12$; Miklavič (2018) for $D \geqslant 9$)
  - the characterization of Γ by $\iota(\Gamma)$ (Gavrilyuk–Koolen (2015) for pseudo-partition graphs; G.–K. (2018) for Grassmann graphs)

- The subclass of bipartite Q-polynomial DRGs is very special ...



- ... in that $\iota(\Gamma)$ determines $\boldsymbol{T}$ algebraically when Γ is bipartite (Caughman, 1999).

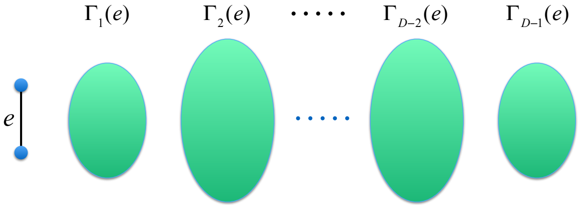
$\mathrm{PSp}_{2D-2}(\mathbb{F}_q) \times \mathfrak{S}_2$ $\qquad$ $\mathrm{PGO}^{+}_{2D}(\mathbb{F}_q)$

- The Hemmeter graphs and the bipartite dual polar graphs are non-isomorphic but share the same $\iota(\Gamma)$, so $\boldsymbol{T}$ cannot distinguish these two families!!

We will further assume that Γ is bipartite!!

... and apply our general theory to this case.

- Fix a base edge $e \in R$.



- The i^{th} dual idempotent $E_i^* = E_i^*(e) \in \mathbb{C}^{X \times X}$ is

$$(E_i^*)_{y,z} = \begin{cases} 1 & \text{if } y = z \in \Gamma_i(e), \\ 0 & \text{otherwise,} \end{cases} \qquad (y, z \in X).$$

cf. [Suzuki (2005)]

- $\boldsymbol{T} = \boldsymbol{T}(e) = \mathbb{C}[A, E_0^*, \ldots, E_{D-1}^*]$: the Terwilliger algebra of Γ w.r.t. e

- Recall

$$\boldsymbol{T} = \boldsymbol{T}(e) = \mathbb{C}[A, E_0^*, \ldots, E_{D-1}^*].$$

- Define the dual adjacency matrix $A^* = A^*(e) \in \mathbb{C}^{X \times X}$ by

$$A^* = A^*(e) = \frac{1}{2} \sum_{x \in e} A^*(x).$$

Theorem

- $\boldsymbol{T} = \mathbb{C}[A, A^*]$, *with two exceptions for each* $D \geqslant 3$.

Corollary

- A, A^* *act on irreducible* $\boldsymbol{T}$*-modules as tridiagonal pairs, unless* Γ *is one of the two exceptions.*

Remark

- $T(e)$ distinguishes the Hemmeter graphs and the bipartite dual polar graphs!!

non-edge-transitive (Hemmeter graphs) edge-transitive (bipartite dual polar graphs)

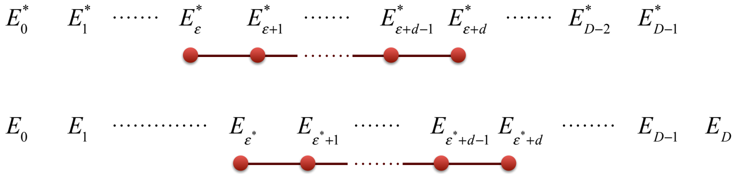
- W : an irreducible $\boldsymbol{T}$-module

Lemma

- $\exists\, \varepsilon, \varepsilon^*, d$ s.t.

$$\{i : E_i^* W \neq 0\} = \{\varepsilon, \varepsilon+1, \ldots, \varepsilon+d\},$$
$$\{\ell : E_\ell W \neq 0\} = \{\varepsilon^*, \varepsilon^*+1, \ldots, \varepsilon^*+d\}.$$

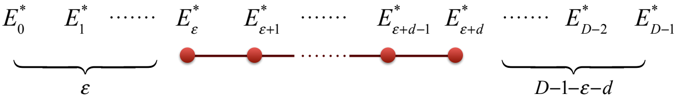
- We may use the following diagrams:



Theorem

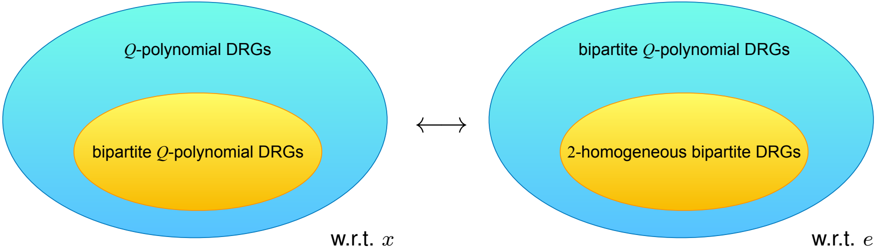
- $2\varepsilon + d \geqslant D - 1$ $\iff$ $\varepsilon \geqslant D - 1 - \varepsilon - d$
- "$=$" $\Longrightarrow$ ① W : *thin, i.e.,* $\dim$ ● $= 1$ *for all* ●. $\longleftrightarrow$ evaluation module

  ② *The structure of* W *is determined by* $\iota(\Gamma)$, ε, *and* ε^*.



- W : short $\overset{\text{def}}{\iff}$ $2\varepsilon + d = D - 1$

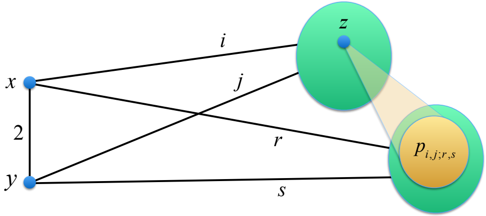
- We claim that the subclass of 2-homogeneous bipartite DRGs is very special:



- Γ : 2-homogeneous

$\overset{\text{def}}{\Longleftrightarrow} \exists p_{i,j;r,s}$ s.t. $\forall x, y \in X$ with $\partial(x,y) = 2$ & $\forall z \in \Gamma_i(x) \cap \Gamma_j(y)$:

$$|\Gamma(z) \cap \Gamma_r(x) \cap \Gamma_s(y)| = p_{i,j;r,s}$$



Theorem (Curtin (1998))

- *TFAE:*
  1. Γ *: 2-homogeneous*
  2. Γ *: Q-polynomial and antipodal (double cover)*

$|\Gamma_D(x)| = 1$

Theorem (Nomura (1995))

- *If Γ is 2-homogeneous then Γ is one of the following:*
  - *the D-cube $\mathcal{Q}_D$,*
  - *$K_{k,k}$ minus a perfect matching $(D = 3)$,*
  - *a Hadamard graph $(D = 4)$,*
  - *$D = 5$ and*

$$(c_1, c_2, c_3, c_4, c_5) = (1, \mu, k - \mu, k - 1, k), \quad b_i = c_{5-i} \ \ (0 \leqslant i \leqslant 4),$$

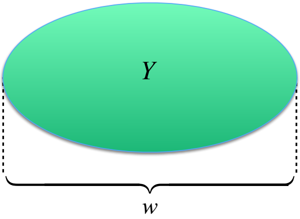
where $k = t(t^2 + 3t + 1)$, $\mu = t(t + 1)$, and $2 \leqslant t \in \mathbb{Z}$.

Theorem

- *Suppose Γ is 2-homogeneous.*
- *Then $\iota(\Gamma)$ determines $\boldsymbol{T}(e)$ algebraically.*
- *Moreover, every irreducible $\boldsymbol{T}(e)$-module is short.*

Consider a general Q-polynomial DRG Γ.

- $Y \subset X$: a non-empty subset of X
- $\chi \in \mathbb{C}^X$: the characteristic vector of Y
- $w = \max\{i : \chi^{\mathsf{T}} A_i \chi \neq 0\}$: the width of Y
- $w^* = \max\{\ell : \chi^{\mathsf{T}} E_\ell \chi \neq 0\}$: the dual width of Y



Theorem (Brouwer–Godsil–Koolen–Martin (2003))

- $w + w^* \geqslant D$.

Theorem

- "$=$" $\Longrightarrow$ *The subgraph induced on Y is a Q-polynomial DRG with diameter w, with at most three exceptions for each $D \geqslant 3$.*

- Y : a descendent of $\Gamma \overset{\text{def}}{\Longleftrightarrow} w + w^* = D$

Example

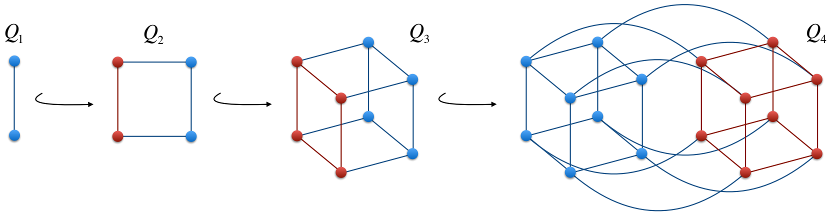
- $Y = \{x\}$: a descendent with $w = 0 \quad (\forall x \in X)$

Example

- Γ : bipartite
- $Y = e \in R$: a descendent with $w = 1$

Example

- $\Gamma = \mathcal{Q}_D$
- $\mathcal{Q}_i \subset \mathcal{Q}_D$ (an i-dimensional face) : a descendent with $w = i$



- Y : a descendent of Γ
- Define $\boldsymbol{T} = \boldsymbol{T}(Y)$ similarly.
- Define the dual adjacency matrix $A^* = A^*(Y) \in \mathbb{C}^{X \times X}$ by

$$A^* = A^*(Y) = \frac{1}{|Y|} \sum_{x \in Y} A^*(x).$$

Theorem

- $\boldsymbol{T} = \mathbb{C}[A, A^*]$, *with at most three exceptions for each* $D \geqslant 3$.

Corollary

- A, A^* *act on irreducible* $\boldsymbol{T}$*-modules as tridiagonal pairs, unless* Γ *is one of the exceptions.*

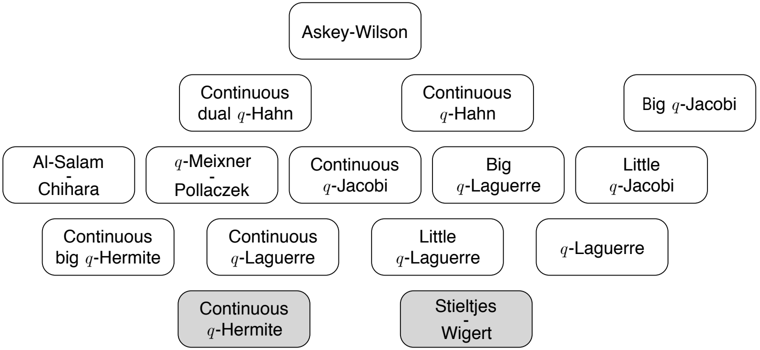
Remark

- $Z \subset Y \subset X$: two descendents of Γ with $w(Y) = w(Z) + 1$
- We can capture (part of) the non-symmetric Askey scheme in terms of $\mathbb{C}[A, A^*(Y), A^*(Z)]$ (cf. Lee (2017); Lee–T. (2018)).

$w(Z)=0$
$w(Y)=1$

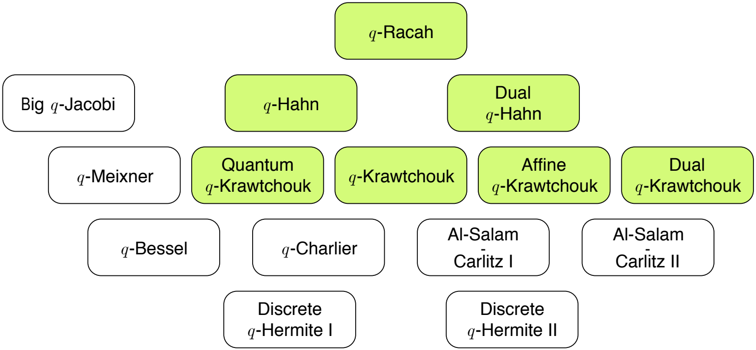
Askey scheme

- q-Hypergeometric orthogonal polynomials, part 1[‡]



‡ taken from: R. Koekoek, P. A. Lesky, and R. F. Swarttouw, Hypergeometric orthogonal polynomials and their q-analogues, Springer-Verlag, Berlin, 2010.

- q-Hypergeometric orthogonal polynomials, part 2[‡]



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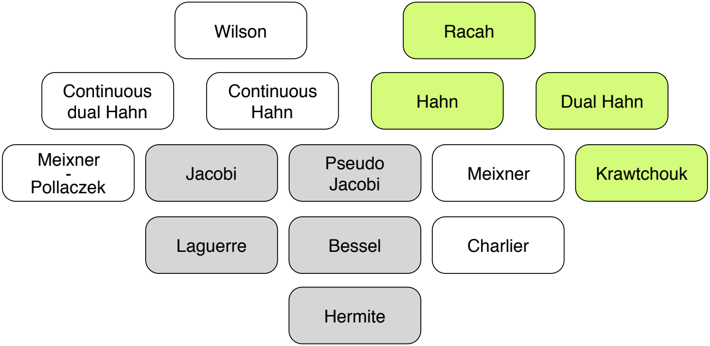
Askey scheme

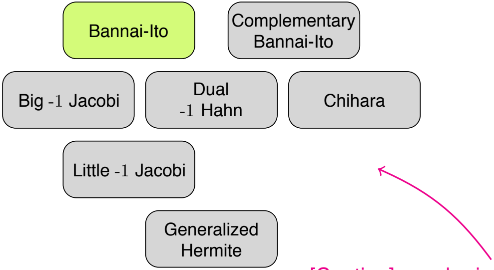
- Hypergeometric orthogonal polynomials[‡]



‡ taken from: R. Koekoek, P. A. Lesky, and R. F. Swarttouw, Hypergeometric orthogonal polynomials and their q-analogues, Springer-Verlag, Berlin, 2010.

Askey scheme

- (-1)-Hypergeometric orthogonal polynomials



[Caution] may be incomplete or wrong!

Remark

The polynomials with $q = -1$ have recently been actively studied by Alexei Zhedanov and others.

Jump to 2

- $V = \mathbb{C}^n$
- $A, A^* \in \mathbb{C}^{n \times n}$: diagonalizable
- $V_0, V_1, \ldots, V_d$: an ordering of the eigenspaces of A
- $V_0^*, V_1^*, \ldots, V_{d^*}^*$: an ordering of the eigenspaces of A^*

- A, A^* : a tridiagonal pair

$\overset{\text{def}}{\iff}$
- $AV_i^* \subset V_{i-1}^* + V_i^* + V_{i+1}^* \quad (0 \leqslant i \leqslant d^*)$,
- $A^*V_i \subset V_{i-1} + V_i + V_{i+1} \quad (0 \leqslant i \leqslant d)$,
- V is irreducible as a $\mathbb{C}[A, A^*]$-module,

where $V_{-1} = V_{d+1} = V_{-1}^* = V_{d^*+1}^* := 0$.

Go to 13