Tight relative t-designs on two shells in hypercubes, and Hahn and Hermite polynomials

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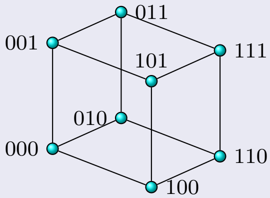
Relative t-designs in the n-cube $\mathcal{Q}_n$

- $[n] := \{1, 2, \ldots, n\}$ $(n \in \mathbb{N})$
- $\mathcal{Q}_n := 2^{[n]} = \{x : x \subseteq [n]\}$: the n-cube
- $\binom{[n]}{k} = \{x \in 2^{[n]} : |x| = k\}$
- $\emptyset \neq Y \subset 2^{[n]}$
- $\omega : Y \to \mathbb{R}_{>0}$

- $\{0,1\}^n \overset{1:1}{\longleftrightarrow} 2^{[n]}$
  - $1010\cdots0 \longleftrightarrow \{1,3\}$
  - $1110\cdots0 \longleftrightarrow \{1,2,3\}$



Definition (Delsarte (1977))

- (Y, ω) : a relative t-design ← "weighted" regular t-wise balanced design

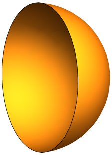
$\overset{\text{def}}{\Longleftrightarrow} \exists \lambda_1, \lambda_2, \ldots, \lambda_t \in \mathbb{R}_{>0}$ s.t. for $i = 1, 2, \ldots, t$,

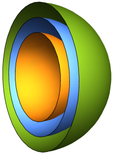
$$\sum_{z \subset y \in Y} \omega(y) = \lambda_i \qquad \left(\forall z \in \binom{[n]}{i}\right)$$

Delsarte's design theory

- $\Phi := \left\{k : Y \cap \binom{[n]}{k} \neq \emptyset\right\}$: the set of block sizes of Y

$\lvert\Phi\rvert = 1$ (t-designs) Delsarte (1973)	$\longrightarrow$	$\lvert\Phi\rvert \geqslant 2$ (relative t-designs) Delsarte (1977)
$\updownarrow$		$\updownarrow$
spherical t-designs Delsarte–Goethals–Seidel (1977)	$\longrightarrow$	Euclidean t-designs Neumaier–Seidel (1988)





Tight relative t-designs

- Recall $\Phi = \left\{k : Y \cap \binom{[n]}{k} \neq \emptyset\right\}$.

Theorem (Bannai–Bannai (2012); Xiang (2012))

- (Y, ω) *: a relative* $2e$*-design*
- $e \leqslant k \leqslant n - e \quad (\forall k \in \Phi)$
- *Then*

$$|Y| \geqslant \binom{n}{e} + \binom{n}{e-1} + \cdots + \binom{n}{e-|\Phi|+1}.$$

Fisher-type inequality

Definition

- (Y, ω) : tight $\overset{\text{def}}{\Longleftrightarrow}$ $|Y| = \binom{n}{e} + \binom{n}{e-1} + \cdots + \binom{n}{e-|\Phi|+1}$

The tight case with $|\Phi| = 1$

Theorem (Delsarte (1973), Wilson–Ray-Chaudhuri (1975))

- $\Phi = \{k\}$ *where* $e \leqslant k \leqslant n/2$
- (Y, ω) *: a tight* $2e$*-design* $\subset \binom{[n]}{k}$
- *Then*
  1. Y *induces an* e*-class* Q*-polynomial association scheme.*
  2. *The polynomial*

$$\psi_e^k(\xi) := {}_3F_2\left(\begin{matrix} -\xi, -e, e-n \\ k-n+1, 1-k \end{matrix} \middle| 1\right) \in \mathbb{R}[\xi]$$

of degree e *has only integral zeros.*

take complement: $k \leftrightarrow n-k$

the Wilson polynomial

The tight case with $|\Phi| = 1$

Remark

- Infinitely many for $e = 1$
- Ito (1975), Bremner (1979): only 2 examples for $e = 2$
- Peterson (1977): none for $e = 3$
- Bannai (1977): only finitely many, for each $e \geqslant 5$
- Dukes–Short-Gershman (2013): none for $e = 5, 6, 7, 8, 9$
- Xiang (2018): none for $e = 4$

The Hahn polynomials

- $\alpha, \beta \in \mathbb{R}, \ \ N \in \mathbb{N}$
- The Hahn polynomial of degree e $(e = 0, 1, \ldots, N)$ is

$$Q_e(\xi; \alpha, \beta, N) = {}_3F_2\left(\left. \begin{matrix} -\xi, -e, e+\alpha+\beta+1 \\ \alpha+1, -N \end{matrix} \right| 1 \right) \in \mathbb{R}[\xi].$$

Remark

- $\psi_e^k(\xi) = Q_e(\xi; k-n, -k-1, k-1)$

The tight case with $|\Phi| = 2$

take complement: $\ell \leftrightarrow n-\ell$, $m \leftrightarrow n-m$

Theorem

- $\Phi = \{\ell, m\}$ *where* $e \leqslant \ell < m \leqslant n-\ell$
- (Y, ω) *: a tight relative* $2e$*-design* $\subset \binom{[n]}{\ell} \sqcup \binom{[n]}{m}$
- *Then*
  1. Y *induces a coherent configuration of type* $\left[\begin{smallmatrix} e+1 & e \\ & e+1 \end{smallmatrix}\right]$.
  2. *The polynomial*

$$\psi_e^{\ell,m}(\xi) := {}_3F_2\left(\left.\begin{matrix} -\xi, -e, e-n-1 \\ m-n, -\ell \end{matrix}\right| 1\right) \in \mathbb{R}[\xi]$$

     of degree e *has only integral zeros.*

Remark

- $\psi_e^{\ell,m}(\xi) = Q_e(\xi; m-n-1, -m-1, \ell)$

A tool in the proof

$|\Phi| = 1$ (t-designs)
Bose–Mesner algebra
(commutative)

→

$|\Phi| = 2$ (relative t-designs)
Terwilliger algebra
(non-commutative)

Use of $\psi_e^{\ell,m}(\xi)$

Example

- Bannai–Bannai–Zhu (2017) found two tight relative 4-designs:

n	ℓ	m	ξ
22	6	7	$3, 5$
22	6	15	$1, 3$

- The zeros ξ are integers!!

Use of $\psi_e^{\ell,m}(\xi)$

Example

- The existence of tight relative 4-designs with the following feasible parameters were left open:

n	ℓ	m	ξ
37	9	16	$\frac{1}{14}(71 \pm \sqrt{337})$
37	9	21	$\frac{1}{14}(55 \pm \sqrt{337})$
41	15	16	$\frac{1}{26}(237 \pm \sqrt{1569})$
41	15	25	$\frac{1}{26}(153 \pm \sqrt{1569})$

- The zeros ξ are irrational, thus proving the non-existence!!

Bannai's result revisited ($|\Phi| = 1$)

Theorem (Bannai (1977))

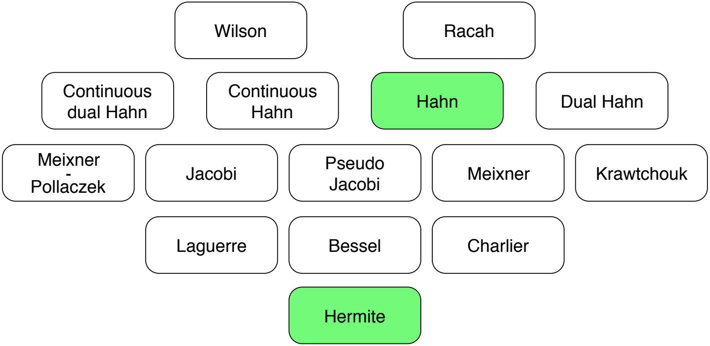
- *Fix $e \geqslant 5$.*
- *Then only finitely many non-trivial tight $2e$-designs.*

- The Hermite polynomial of degree e ($e = 0, 1, 2, \ldots$) is

$$H_e(\eta) = (2\eta)^e \cdot {}_2F_0\left(\left. \begin{matrix} -e/2, -(e-1)/2 \\ - \end{matrix} \right| -\frac{1}{\eta^2} \right) \in \mathbb{R}[\eta].$$

- $Q_e(\xi; \alpha, \beta, N) \approx H_e(\eta)$ for appropriate limit $|\alpha|, |\beta|, N \to \infty$
- Bannai applied this to $\psi_e^k(\xi) = Q_e(\xi; k-n, -k-1, k-1)$.

- Hypergeometric orthogonal polynomials[1]



[1] taken from: R. Koekoek, P. A. Lesky, and R. F. Swarttouw, Hypergeometric orthogonal polynomials and their q-analogues, Springer-Verlag, Berlin, 2010.

A non-existence result for the case $|\Phi| = 2$

- Recall $\Phi = \{\ell, m\}$ $(e \leqslant \ell < m \leqslant n - \ell)$.

Theorem

- $\forall \delta \in (0,1)$ $\exists e_0 = e_0(\delta) > 0$ *such that:*
  1. *Fix* $e \geqslant e_0$ *and* $c > 0$.
  2. *Then only finitely many such tight relative* $2e$*-designs with*

$$\ell < c \cdot n^{1-\delta}.$$

up to scalar multiple of ω

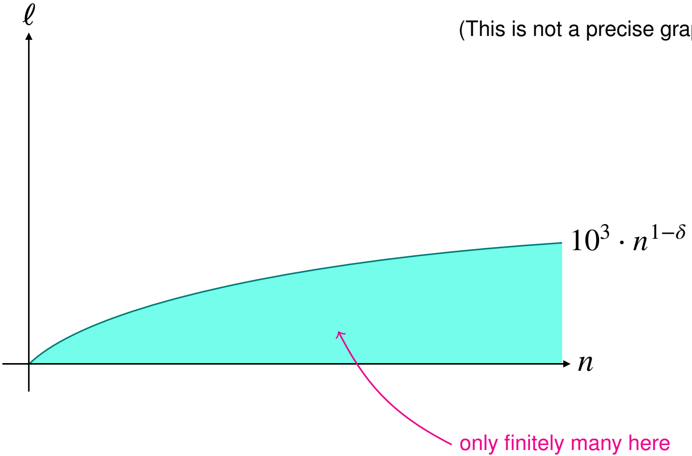
Remark

- We can take $e_0(\delta) = \exp(2.51012/\delta)$.

A non-existence result for the case $|\Phi| = 2$

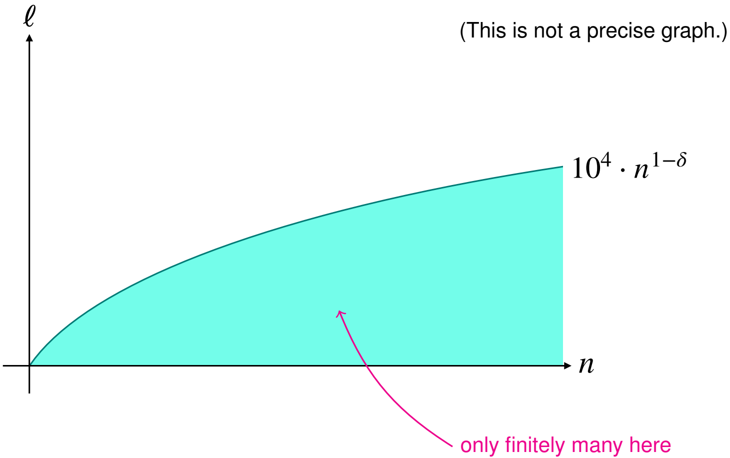
- Fix $e \geqslant e_0(\delta)$ and set $c = 1,000 = 10^3$.

(This is not a precise graph.)



A non-existence result for the case $|\Phi| = 2$

- Fix $e \geqslant e_0(\delta)$ and set $c = 10,000 = 10^4$.



A non-existence result for the case $|\Phi| = 2$

- Fix $e \geqslant e_0(\delta)$ and set $c = 100,000 = 10^5$.