Scaling limits for the Gibbs states on distance-regular graphs with classical parameters

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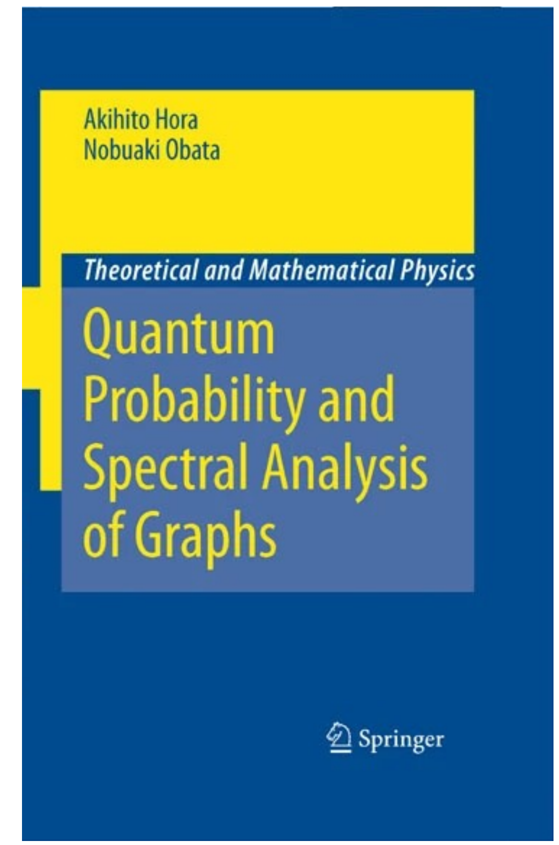
19th Workshop: Noncommutative probability, noncommutative harmonic analysis and related topics

Today's topic

- To obtain CLT-type theorems for algebraic probability spaces arising from certain graphs

  "distance-regular"

Reference: A. Hora & N. Obata, *Quantum Probability and Spectral Analysis of Graphs*, Springer-Verlag, 2007.



① I specialize in DRGs.

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② I attended a talk given by Obata in 2009.

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③ I (and some other participants) quickly found how it is related to the theory of DRGs (i.e., the Terwilliger algebra).

Algebraic probability spaces

- $(\mathscr{A}, \varphi)$: an algebraic probability space
- $\forall a \in \mathscr{A}$: called an **algebraic random variable**
- $a \in \mathscr{A}$: **real** $\overset{\text{def}}{\iff} a^* = a$

Remark. For every real $a \in \mathscr{A}$, there exists a Borel probability measure μ on $\mathbb{R}$ s.t.

i^{th} moment

$$\varphi(a^i) = \int_{-\infty}^{+\infty} \xi^i \, \mu(d\xi) \quad (i = 0,1,2,\ldots).$$

Orthogonal polynomials

- μ : a Borel probability measure on $\mathbb{R}$ with finite moments
- $p_0, p_1, p_2, \ldots$: the **monic orthogonal polynomials** w.r.t.

$$(f, g)_\mu = \int_{-\infty}^{+\infty} \overline{f(\xi)} g(\xi) \mu(d\xi) \quad (f, g \in \mathbb{C}[\xi]).$$

three-term recurrence

Remark. $\exists \omega_i > 0$, $\exists \alpha_i$ $(i = 1,2,3,\ldots)$ s.t.

$$\xi p_i(\xi) = p_{i+1}(\xi) + \alpha_{i+1} p_i(\xi) + \omega_i p_{i-1}(\xi) \quad (i = 0,1,\ldots),$$

where $p_{-1}(\xi) = 0$, and ω_0 is undefined.

$$\xi p_i = p_{i+1} + \alpha_{i+1} p_i + \omega_i p_{i-1}$$

Remark. If $d+1 = |\operatorname{supp}\mu| < \infty$, then we only have $p_0, \ldots, p_d$, and thus only have $\omega_1, \ldots, \omega_d$ and $\alpha_1, \ldots, \alpha_{d+1}$.

Jacobi coefficients

Remark. The scalars ω_i and α_i conversely determine

$$\int_{-\infty}^{+\infty} \xi^i \, \mu(d\xi) \quad (i = 0,1,2,\ldots)$$

i^{th} moment

by the **Accardi–Bożejko formula**.

$$(\mathscr{A}, \varphi) \ni a \longmapsto \mu \longmapsto \{\omega_i\},\ \{\alpha_i\}$$

real

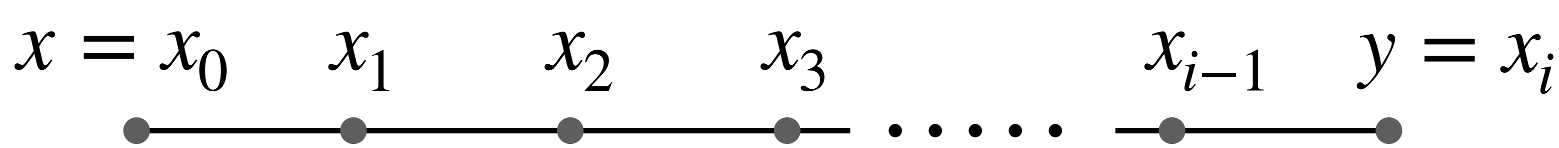
finite moments

Graphs

vertex set

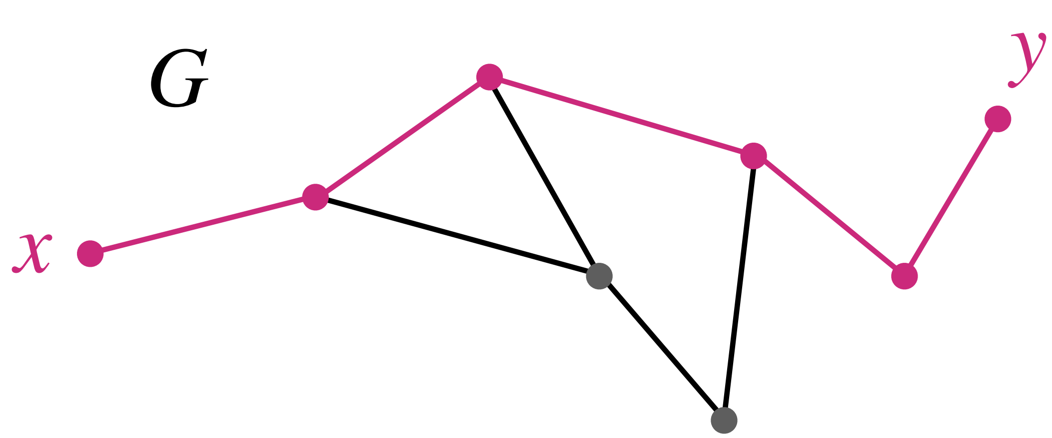
edge set $=$ a set of 2-element subsets of X

- $G = (V, E)$: a finite connected simple graph
- ∂ : the **path-length distance** on V :



$$\partial(x, y) = i$$

- $d := \max\{\partial(x, y) : x, y \in V\}$: the **diameter** of G

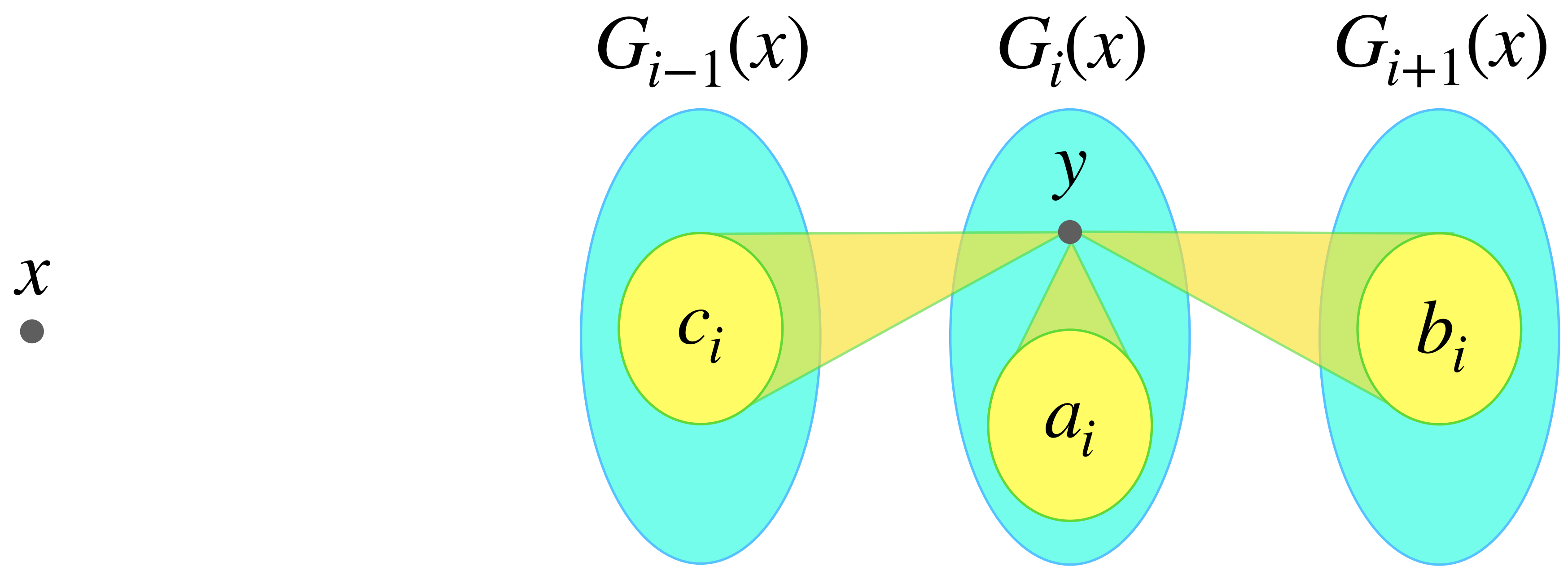


$d = 5$

Distance-regular graphs

- $G = (V, E)$: a finite connected simple graph with diameter d
- $G_i(x) = \{y : \partial(x, y) = i\}$: the i^{th} **subconstituent** w.r.t. x
- G : **distance-regular**

$\overset{\text{def}}{\iff} \exists a_i, b_i, c_i \ (i = 0, \ldots, d)$ s.t. $\forall x, y \in V$ with $\partial(x, y) = i$:



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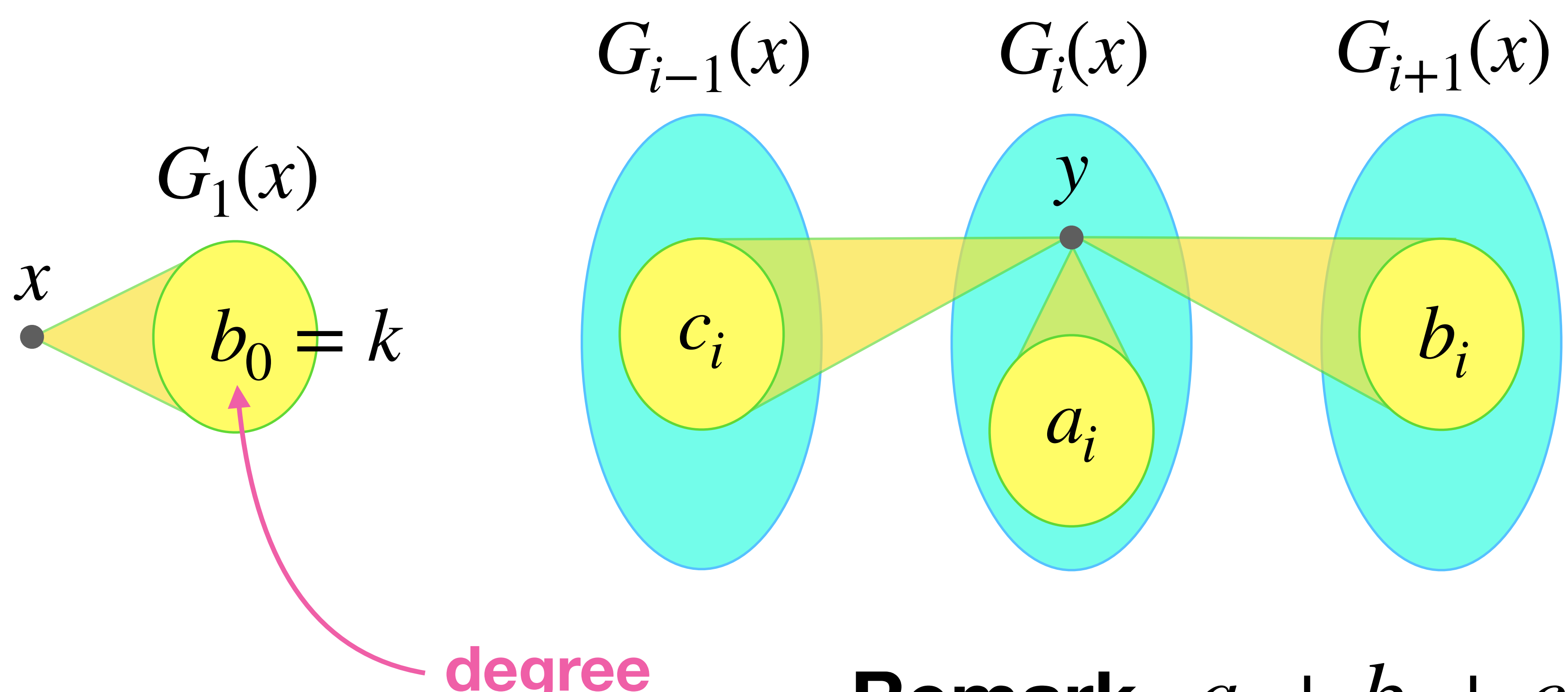
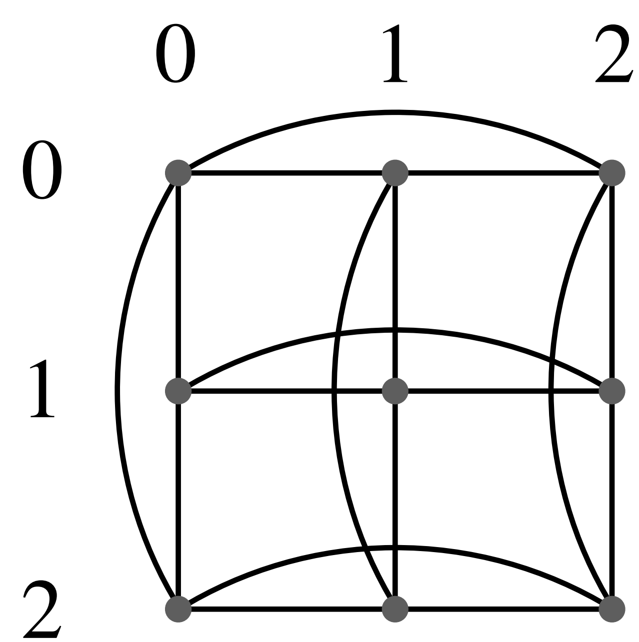


Remark. $a_i + b_i + c_i = k = b_0$

Example (Hamming graphs $H(d, n)$).

- $V = \{0,1,\ldots,n-1\}^d$
  $= \left\{(x_1, x_2, \ldots, x_d) : x_1, x_2, \ldots, x_d \in \{0,1,\ldots,n-1\}\right\}$
- $x = (x_\ell) \sim y = (y_\ell) \overset{\text{def}}{\iff} |\{\ell : x_\ell \neq y_\ell\}| = 1$
- $b_i = (d-i)(n-1),\ c_i = i \quad (i = 0,1,\ldots,d)$



$H(2,3)$

- $G = (V, E)$: a distance-regular graph with diameter d
- $A \in M_V(\mathbb{C})$: the **adjacency matrix** of G :

$$A_{x,y} = \begin{cases} 1 & \text{if } x \sim y \\ 0 & \text{otherwise} \end{cases} \qquad (x, y \in V)$$

- $\mathscr{A} = \mathbb{C}[A]$: the **adjacency algebra** of G

$$\varphi_{\mathrm{tr}}(B) = \frac{\mathrm{tr}(B)}{|V|}$$

- Consider the algebraic probability space $(\mathscr{A}, \varphi_{\mathrm{tr}})$, where φ_{tr} denotes the **tracial state**.
- The probability measure corresponding to the real algebraic random variable A is the **spectral distribution** μ_G of A.

"Central Limit Theorem"

Problem. If G "grows", then $\mu_G \to \exists \mu$?

mean

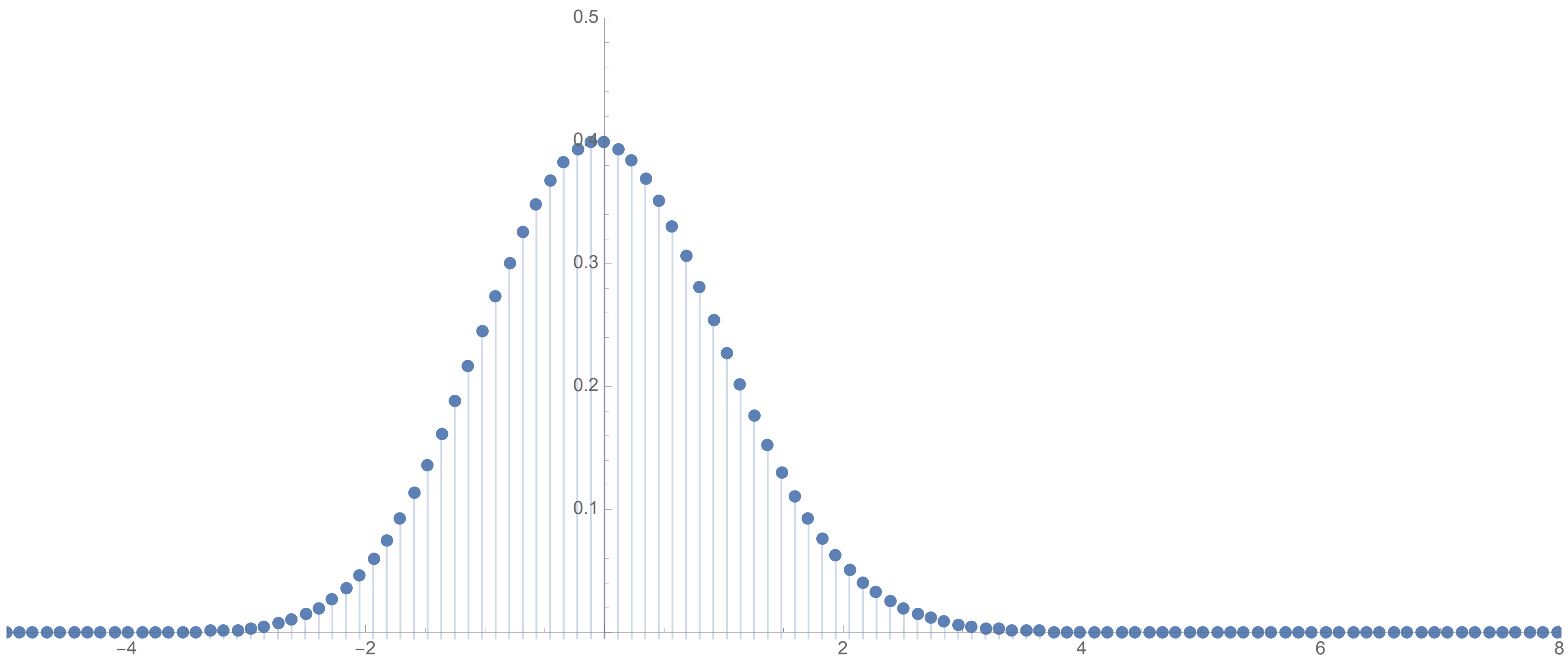
variance

- Since $\varphi_{\mathrm{tr}}(A) = 0$ and $\varphi_{\mathrm{tr}}(A^2) = k$, we will instead work with $A/\sqrt{k}$, and normalize μ_G accordingly.

Example (Hamming graphs $H(d, n)$).

$k = d(n-1)$

- $n/d \to 0$



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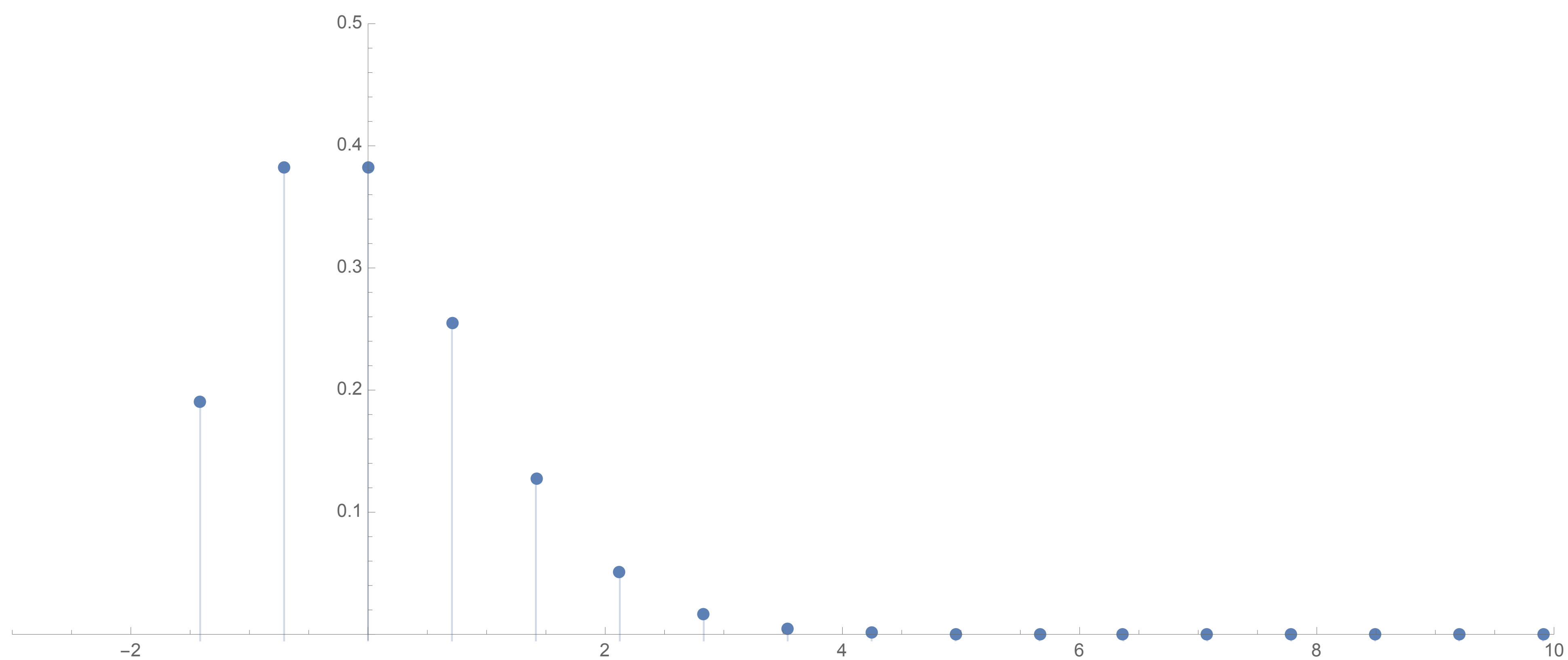

- The limit distributions have been computed by Hora, Obata, and others, for DRGs including Hamming, Johnson, Odd, and Grassmann graphs.

graphs	limit distributions
Hamming	Gaussian, Poisson
Johnson	geometric, exponential

Gibbs states

- Let $t \in \mathbb{R}$, and define $Q_t \in \mathscr{A}$ by

$$(Q_t)_{x,y} = t^{\partial(x,y)} \quad (x, y \in V).$$

- The **Gibbs state** φ_t on $\mathscr{A}$ is defined by

$$\varphi_t(B) = \frac{1}{|V|}\langle Q_t, B\rangle \quad (B \in \mathscr{A}).$$

or **deformed vacuum state**

Remark. $\varphi_0 = \varphi_{\mathrm{tr}}$.

Remark. φ_t : a state $\iff Q_t \succcurlyeq 0$

positive semidefinite

φ_t : a state $\Longleftrightarrow Q_t \geqslant 0$

- $\pi(G) = \{t \in \mathbb{R} : Q_t \geqslant 0\} \subset [-1,1]$
- We have

mean, variance

$$\varphi_t(A) = tk, \qquad \varphi_t\left((A - tkI)^2\right) = \underline{k(1-t)(1+t+ta_1)}.$$

$$\| \quad \Sigma_t^2$$

- Hence we will work with $(A - tkI)/\Sigma_t$.
- Hora ('00) showed $[0,1] \subset \pi(G)$ if G is a Hamming graph or a Johnson graph, and computed the limit distributions:

graphs	limit distributions
Hamming	Gaussian, Poisson
Johnson	compound Poisson distributions of gamma and Pascal distributions

DRGs with classical parameters

- $G = (V, E)$: a DRG with diameter d
- G is said to have **classical parameters** (d, q, α, β) if

$$b_i = \left(\begin{bmatrix} d \\ 1 \end{bmatrix} - \begin{bmatrix} i \\ 1 \end{bmatrix} \right) \left(\beta - \alpha \begin{bmatrix} i \\ 1 \end{bmatrix} \right),$$

$$c_i = \begin{bmatrix} i \\ 1 \end{bmatrix} \left(1 + \alpha \begin{bmatrix} i-1 \\ 1 \end{bmatrix} \right)$$

$$\begin{bmatrix} n \\ 1 \end{bmatrix} = \begin{bmatrix} n \\ 1 \end{bmatrix}_q = 1 + q + \cdots + q^{n-1}$$

for $i = 0, 1, \ldots, d$.

Remark. Most of the known infinite families of DRGs either have classical parameters or are related to such families.

$$b_i = \left(\begin{bmatrix} d \\ 1 \end{bmatrix} - \begin{bmatrix} i \\ 1 \end{bmatrix}\right)\left(\beta - \alpha \begin{bmatrix} i \\ 1 \end{bmatrix}\right)$$
$$c_i = \begin{bmatrix} i \\ 1 \end{bmatrix}\left(1 + \alpha \begin{bmatrix} i-1 \\ 1 \end{bmatrix}\right)$$

Example (Hamming graphs $H(d, n)$).

- $b_i = (d - i)(n - 1),\ c_i = i$
- $q = 1,\ \alpha = 0,\ \beta = n - 1$

$\pi(G) = \{t \in \mathbb{R} : Q_t \succcurlyeq 0\}$

Proposition (Koohestani–Obata–T., '21). *If G has classical parameters with $q \neq 1$, then* $q^{-i} \in \pi(G)$ $(i = 0, 1, 2, \ldots)$.

(cf. Voit, '21)

For each $\lambda \in \Lambda$, fix $t \in \pi(G_\lambda)$.

Theorem (Koohestani–Obata–T., '21)**.** *Assume the following.*

- $(\Lambda, \leqslant)$: *a directed set*
- $(G_\lambda)_{\lambda\in\Lambda}$: *a net of DRGs, where* $d \to \infty$, *such that:*

  ① *Each* G_λ *has classical parameters* (d, q, α, β) *with* $q \neq 1$.

  ② *The limit Jacobi coefficients of* $(A - tkI)/\Sigma_t$ *exist.*

Then q *eventually takes at most three values. Suppose that* q *is eventually constant. Then so is* α, *and the following hold:*

- *If* $\alpha \neq 0$, *then* $\beta/\sqrt{k}$ *is eventually bounded, and* $\exists \gamma, \rho \in \mathbb{R}$ *s.t.* $\rho > 0$, $\gamma(\rho + \alpha/\rho) > -1$, $t\sqrt{k} \to \gamma$, *and the accumulation points of* $\beta/\sqrt{k}$ *are in* $\{\rho, \alpha/\rho\}$.
- *If* $\alpha = 0$, *then* $\exists \gamma, \rho \in \mathbb{R}$ *s.t.* $\rho \geqslant 0$, $\gamma\rho > -1$, $t\sqrt{k} \to \gamma$, *and* $\beta/\sqrt{k} \to \rho$.

- $\sharp q \leqslant 3$
- If $\sharp q = 1$ then
  - $t\sqrt{k} \to \gamma$
  - $\beta/\sqrt{k} \to \{\rho, \alpha/\rho\}$

Remark. If q is not constant, then there exists a subnet of $(G_\lambda)_{\lambda\in\Lambda}$ for which q is eventually constant and $\alpha = 0$.

Remark. Many of the previous results are sufficient conditions for the existence of limit distributions. Our theorem provides a necessary condition, which is also more or less sufficient.

Remark. The limit distributions are explicitly described in terms of q, α, γ, ρ (and one other parameter when $\alpha = 0$).

Remark. For $\gamma = 0$, the corresponding orthogonal polynomials belong to the **Askey scheme** of q-hypergeometric orthogonal polynomials.