2025年8月 (August 2025)

1 Let $D = \left\{(x, y) \in \mathbb{R}^2 \mid -1 \leq 2x+1 \leq x+y \leq 0\right\}$. Answer the following questions.

(1) Draw a picture of the region D in the xy-plane.

(2) Find the set of points in the region $\left\{(r, \theta) \in \mathbb{R}^2 \mid r > 0,\ 0 \leq \theta < 2\pi\right\}$ that are mapped to D by the two-dimensional polar coordinate transformation.

(3) Evaluate the double integral

$$\iint_D \frac{1}{(x^2+y^2)^{3/2}} \arctan \frac{y}{x}\, dxdy,$$

where arctan denotes the inverse of $\tan : \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \to \mathbb{R}$.

2 Let c be a real number. For the two matrices

$$A = \begin{pmatrix} 1 & 2 & 0 & 0 \\ 2 & 4 & 0 & 0 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & c & c \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 2 & 0 & 4 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 2 \end{pmatrix},$$

we define the linear mappings $f_A, f_B \colon \mathbb{R}^4 \to \mathbb{R}^4$ by $f_A(\boldsymbol{x}) = A\boldsymbol{x}$ and $f_B(\boldsymbol{x}) = B\boldsymbol{x}$, where $\boldsymbol{x}$ denotes a column vector in $\mathbb{R}^4$.

(1) Find a basis of the kernel $\operatorname{Ker} f_A$ of f_A.

(2) Find a basis of the kernel $\operatorname{Ker} f_B$ of f_B.

(3) Suppose that the intersection of the image $f_B(\operatorname{Ker} f_A)$ of $\operatorname{Ker} f_A$ under f_B and the image $f_A(\operatorname{Ker} f_B)$ of $\operatorname{Ker} f_B$ under f_A contains a nonzero vector. Find the real number c.

3 $\mathbb{R}^2$ 上の 2 変数関数 $f(x, y) = (x-y)(|x|+|y|+1)$ を考える.

(1) 点 $(0, 0)$ において $f(x, y)$ が偏微分可能であることを示せ.

(2) 点 $(0, 0)$ において $f(x, y)$ が全微分可能であることを示せ.

(3) $f(x, y)$ は $\mathbb{R}^2$ 上で C^1 級とはならないことを示せ.

$\boxed{3}$ Consider the two-variable function $f(x,y) = (x-y)(|x|+|y|+1)$ on $\mathbb{R}^2$.

(1) Show that $f(x,y)$ is partially differentiable at $(x,y) = (0,0)$.

(2) Show that $f(x,y)$ is totally differentiable at $(x,y) = (0,0)$.

(3) Prove that $f(x,y)$ is not of class C^1 on $\mathbb{R}^2$.

$\boxed{4}$ $A = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}$ とする．ベクトル空間 $\mathbb{R}^4$ の列ベクトル $\boldsymbol{u}, \boldsymbol{v}$ に対して，標準的な内積を $(\boldsymbol{u}, \boldsymbol{v})$ で表す．

(1) A の固有値と固有ベクトルを求めよ．

(2) $\boldsymbol{u}$ が零ベクトルでない $\mathbb{R}^4$ の列ベクトル全体を動くとき，$\dfrac{(\boldsymbol{u}, A\boldsymbol{u})}{(\boldsymbol{u}, \boldsymbol{u})}$ の最大値を求めよ．

(3) $\mathbb{R}^4$ の 2 つの列ベクトル $\boldsymbol{w} = \begin{pmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \end{pmatrix}$, $\boldsymbol{w}' = \begin{pmatrix} w_1' \\ w_2' \\ w_3' \\ w_4' \end{pmatrix}$ は，

$$(\boldsymbol{w}, \boldsymbol{w}) = (\boldsymbol{w}', \boldsymbol{w}') = 1, \qquad (\boldsymbol{w}, \boldsymbol{w}') = 0$$

を満たすとする．4×2 行列 S を

$$S = \begin{pmatrix} \boldsymbol{w} & \boldsymbol{w}' \end{pmatrix} = \begin{pmatrix} w_1 & w_1' \\ w_2 & w_2' \\ w_3 & w_3' \\ w_4 & w_4' \end{pmatrix}$$

とし，$B = {}^tSAS$ とおく．ここで，${}^tS$ は S の転置行列を表す．B の固有値の最大値が A の固有値の最大値を超えないことを示せ．

$\boxed{4}$ Let $A = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}$. For column vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ in the vector space $\mathbb{R}^4$, their standard inner product is denoted by $(\boldsymbol{u}, \boldsymbol{v})$.

(1) Find the eigenvalues and the eigenvectors of A.

(2) Find the maximum value of $\dfrac{(\boldsymbol{u}, A\boldsymbol{u})}{(\boldsymbol{u}, \boldsymbol{u})}$ when $\boldsymbol{u}$ varies over the nonzero column vectors in $\mathbb{R}^4$.

(3) Assume that two column vectors $\boldsymbol{w} = \begin{pmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \end{pmatrix}$ and $\boldsymbol{w}' = \begin{pmatrix} w_1' \\ w_2' \\ w_3' \\ w_4' \end{pmatrix}$ in $\mathbb{R}^4$ satisfy

$$(\boldsymbol{w}, \boldsymbol{w}) = (\boldsymbol{w}', \boldsymbol{w}') = 1 \quad \text{and} \quad (\boldsymbol{w}, \boldsymbol{w}') = 0.$$

Define a 4×2 matrix S by

$$S = \begin{pmatrix} \boldsymbol{w} & \boldsymbol{w}' \end{pmatrix} = \begin{pmatrix} w_1 & w_1' \\ w_2 & w_2' \\ w_3 & w_3' \\ w_4 & w_4' \end{pmatrix},$$

and let $B = {}^tSAS$. Here, ${}^tS$ denotes the transpose of S. Show that the maximum of the eigenvalues of B does not exceed the maximum of the eigenvalues of A.