2026年8月 (August 2026)

1 Let a and b be real numbers such that $a < b$. Let $f(x)$ be a real-valued continuous function on the closed interval $[a, b]$. Assume that $f(a) > a$ and $f(b) < b$.

(1) Show that there exists $x \in (a, b)$ such that $f(x) = x$.

(2) Show that the set $A = \{x \in (a, b) \mid f(x) = x\}$ has the maximum element and the minimum element.

(3) Suppose that $f(x)$ is a monotonically increasing function. Show that the sequence $\{a_n\}$ defined by

$$a_1 = a \qquad \text{and} \qquad a_{n+1} = \frac{f(a_n) + a_n}{2} \quad (n = 1, 2, 3, \ldots)$$

converges to the minimum element of the set A given in (2).

2 Let a be a real number. For the matrix

$$A = \begin{pmatrix} a & 1 & 1 & 1 \\ 1 & a & 1 & 1 \\ 1 & 1 & a & 1 \\ 1 & 1 & 1 & a \end{pmatrix},$$

we define the linear mapping $f : \mathbb{R}^4 \to \mathbb{R}^4$ by $f(\boldsymbol{x}) = A\boldsymbol{x}$, where $\boldsymbol{x}$ denotes a column vector in $\mathbb{R}^4$.

(1) Find $a \in \mathbb{R}$ that maximizes the dimension of the kernel $\mathrm{Ker}(f)$ of f.

(2) For the value of a found in (1), find a basis of $\mathrm{Ker}(f)$ and a basis of the image $\mathrm{Im}(f)$ of f.

(3) Let n be a positive integer. Show that a linear mapping $g : \mathbb{R}^n \to \mathbb{R}^n$ is surjective if it is injective.

3 $\mathbb{R}^3$ 上の 3 変数関数 $g(x, y, z) = x^2 - y^2 z$ を考える.

(1) 次の条件

$$g(x, y, z) = g_x(x, y, z) = g_y(x, y, z) = g_z(x, y, z) = 0$$

を満たす $(x, y, z) \in \mathbb{R}^3$ をすべて求めよ. ただし, g_x, g_y, g_z はそれぞれ変数 x, y, z に関する g の偏導関数を表す.

(2) 条件 $g(x, y, z) = 0$ の下での関数

$$f(x, y, z) = (x^2 + y^2)e^{-\left(\frac{1}{4}y^2 + \frac{1}{6}z^3\right)}$$

の最大値と最小値を求めよ. ただし, 最大値と最小値が存在することは認めてよい.

3

Consider the three-variable function $g(x, y, z) = x^2 - y^2 z$ on $\mathbb{R}^3$.

(1) Find all $(x, y, z) \in \mathbb{R}^3$ satisfying the following condition

$$g(x, y, z) = g_x(x, y, z) = g_y(x, y, z) = g_z(x, y, z) = 0,$$

where g_x, g_y, and g_z denote the partial derivatives of g with respect to the variables x, y, and z, respectively.

(2) Find the maximum and minimum values of the function

$$f(x, y, z) = (x^2 + y^2)e^{-\left(\frac{1}{4}y^2 + \frac{1}{6}z^3\right)}$$

under the condition $g(x, y, z) = 0$. Here, you may assume that the maximum and minimum values exist.

4

a を実数とし，$A = \begin{pmatrix} 1 & a & 1 \\ a & 0 & a \\ 1 & a & 1 \end{pmatrix}$ とする．

(1) A の固有値をすべて求めよ．

(2) A を直交行列 P を用いて対角化せよ．このときの直交行列 P も与えよ．

(3) 任意の列ベクトル $\boldsymbol{x} \in \mathbb{R}^3$ に対して ${}^t\boldsymbol{x}A\boldsymbol{x} \geq 0$ となる $a \in \mathbb{R}$ をすべて求めよ．ここで，${}^t\boldsymbol{x}$ は $\boldsymbol{x}$ の転置を表す．

4

Let a be a real number and let $A = \begin{pmatrix} 1 & a & 1 \\ a & 0 & a \\ 1 & a & 1 \end{pmatrix}$.

(1) Find the eigenvalues of A.

(2) Diagonalize A using an orthogonal matrix P, and give the matrix P.

(3) Find all $a \in \mathbb{R}$ such that ${}^t\boldsymbol{x}A\boldsymbol{x} \geq 0$ holds for every column vector $\boldsymbol{x} \in \mathbb{R}^3$, where ${}^t\boldsymbol{x}$ denotes the transpose of $\boldsymbol{x}$.