March, 2011

$\boxed{1}$ Find the shortest distance between points (x, y, z) on the surface $2xy + z^2 = 1$ and the point $(1, 1, 2)$ in the 3-dimensional Euclidean space.

$\boxed{2}$ For each non-negative integer n, set $I_n = \displaystyle\int_0^{\frac{\pi}{4}} (\tan x)^n \, dx$.

(1) Show that $I_n + I_{n+2} = \dfrac{1}{n+1}$ for any non-negative integer n.

(2) Show that $\{I_n\}_{n=0}^{\infty}$ is monotonically decreasing, and find the value of $\displaystyle\lim_{n\to\infty} I_n$.

(3) Find the value of $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}$.

$\boxed{3}$ Let $M(2;\mathbb{R})$ be the vector space of real square matrices of degree 2. For $A \in M(2;\mathbb{R})$, define the map $f : M(2;\mathbb{R}) \to M(2;\mathbb{R})$ by

$$f(X) = AX - XA \quad (X \in M(2;\mathbb{R})).$$

(1) Show that the map f is linear.

(2) Let $E = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $E_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $E_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $E_3 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

Show that $\{E, E_1, E_2, E_3\}$ is a basis of $M(2;\mathbb{R})$.

(3) Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Find the matrix representation of f with respect to the basis in (2).

$\boxed{4}$ Define the 3×3 matrix A by $A = \dfrac{1}{2}\begin{pmatrix} 1 & 2 & -2 \\ 1 & 1 & 1 \\ 1 & -2 & 4 \end{pmatrix}$.

(1) Find all the eigenvalues and eigenvectors of the matrix A.

(2) Find all the 3-dimensional vectors $\boldsymbol{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ such that $\lim_{n \to \infty} A^n \boldsymbol{x}$ exists.

$\boxed{5}$ Let $\mathbb{Z}$ denote the set of integers.

(1) Let A and B be finite subsets of $\mathbb{Z}$. Show that

$$\max\{\min A, \min B\} = \min_{a \in A,\ b \in B} \Big(\max\{a,\ b\} \Big).$$

(2) Let n be a positive integer, and let X be a finite set. For each $i = 1, 2, \ldots, n$, let f_i be a mapping from X to $\mathbb{Z}$. Show that

$$\max_{1 \le i \le n} \Big(\min_{x \in X} f_i(x) \Big) = \min_{x_1, \ldots, x_n \in X} \Big(\max_{1 \le i \le n} f_i(x_i) \Big).$$

$\boxed{6}$ Define a meromorphic function $f(z)$ on the complex plane by

$$f(z) = \frac{3e^{iz} - e^{3iz} - 2}{z^3},$$

where i is the imaginary unit. Let I_R be the complex line integral of $f(z)$ along the semi-circle $C_R : z = Re^{i\theta}$ $(0 \le \theta \le \pi)$ for a positive number R; namely,

$$I_R = \int_{C_R} f(z) dz.$$

(1) Find the order of pole and the residue of the function $f(z)$ at $z = 0$.

(2) Evaluate the limit $\lim_{R \to 0+} I_R$.

(3) Show that $I_R \to 0$ as $R \to +\infty$.

(4) Prove the following formula:

$$\int_0^{+\infty} \left(\frac{\sin x}{x} \right)^3 dx = \frac{3\pi}{8}.$$

$\boxed{7}$ Let X be a random variable obeying the exponential distribution with parameter $\lambda > 0$, i.e.,

$$P(X \leq x) = \int_0^x \lambda e^{-\lambda t} dt, \qquad x > 0,$$

$$P(X \leq 0) = 0.$$

Let X' be a random variable which is independent of X and obeys the same distribution as X. Set

$$Y = X + X', \qquad Z = X - X'.$$

(1) Find the mean value $\mathbf{E}[X]$ and the variance $\mathbf{V}[X]$.

(2) Find the covariance $\mathbf{Cov}(Y, Z) = \mathbf{E}[(Y - \mathbf{E}[Y])(Z - \mathbf{E}[Z])]$.

(3) For $t > 0$, compute the value of $P(Y \leq t,\ Z \leq t)$.

$\boxed{8}$ Let $x = x(t)$, $y = y(t)$ be functions of class $C^1(\mathbb{R})$ and satisfy

$$x'(t) = x(t)y(t) + x(t),\ \ y'(t) = -x(t)y(t) + y(t),\ \ x(0) = y(0) = 1.$$

(1) Show that $x(t) + y(t) = 2e^t$ for any $t \in \mathbb{R}$.

(2) Find a differential equation that is satisfied by $u(t) = e^{-t}x(t)$, and express $x(t)$, $y(t)$ in terms of t.

$\boxed{9}$ Let (X, d) be a compact, non-empty metric space. Suppose that a map $f : X \to X$ satisfies

$$d(f(x), f(y)) < d(x, y)$$

for all $x, y \in X$ with $x \neq y$.

(1) Prove that f is continuous.

(2) Prove that there exists a point $x \in X$ such that $f(x) = x$.

(3) Prove that such a point x as in (2) is unique.

$\boxed{10}$ For each of the following, either give an example of a finite group together with a suitable explanation or prove that no such a finite group exists.

(1) A commutative group which is not cyclic.

(2) Two groups that have the same order but are not isomorphic.

(3) A non-commutative group all of whose elements except the identity have order 2.

(4) A group with a non-normal subgroup of index 2.