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$\boxed{1}$ Let $\mathbb{R}^n$ be the n-dimensional real vector space equipped with the standard inner product $\boldsymbol{u} \cdot \boldsymbol{v}$. For a subspace W of $\mathbb{R}^n$, define its orthogonal complement $W^\perp$ by

$$W^\perp = \{\boldsymbol{u} \in \mathbb{R}^n : \boldsymbol{u} \cdot \boldsymbol{v} = 0 \text{ for all } \boldsymbol{v} \in W\}.$$

(1) Let $\{\boldsymbol{a}_1, \ldots, \boldsymbol{a}_k\}$ be an orthonormal basis of W, and let $\{\boldsymbol{b}_1, \ldots, \boldsymbol{b}_l\}$ be an orthonormal basis of $W^\perp$. Prove that $\{\boldsymbol{a}_1, \ldots, \boldsymbol{a}_k, \boldsymbol{b}_1, \ldots, \boldsymbol{b}_l\}$ is an orthonormal basis of $\mathbb{R}^n$, and show that $\dim W^\perp = n - \dim W$ holds.
(2) Prove that $(W^\perp)^\perp = W$ holds for any subspace W.
(3) Prove that $(W_1 + W_2)^\perp = W_1^\perp \cap W_2^\perp$ holds for any subspaces W_1, W_2.
(4) Prove that $(W_1 \cap W_2)^\perp = W_1^\perp + W_2^\perp$ holds for any subspaces W_1, W_2.

$\boxed{2}$ Let V denote the vector space consisting of all polynomials of degree at most 3 with real coefficients:

$$V = \{f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 : a_0, a_1, a_2, a_3 \in \mathbb{R}\}.$$

Let W be a subset of V defined by

$$W = \left\{ f(x) \in V : \int_0^1 f(x)dx = 0 \right\}.$$

For $a, b \in \mathbb{R}$, define a linear map $T : V \longrightarrow V$ by

$$(Tf)(x) = f(ax + b).$$

(1) Prove that W is a linear subspace of V.
(2) Find a basis of W.
(3) Assume that $(a, b) \neq (1, 0)$. Determine a, b so that the restriction $T|_W$ to W of T is a linear map from W to W.
(4) For the constants a, b determined in (3), calculate the matrix representation of $T|_W$ with respect to the basis found in (2).

$\boxed{3}$ Let p, q be positive numbers satisfying $\dfrac{1}{p} + \dfrac{1}{q} = 1$.

(1) Let A, B be positive numbers. Show that

$$A^{\frac{1}{p}} B^{\frac{1}{q}} \leq \frac{A}{p} t^p + \frac{B}{q} t^{-q}$$

holds for any positive number t. For which t does equality hold?

(2) Show that

$$x_1^{\frac{1}{p}} y_1^{\frac{1}{q}} + x_2^{\frac{1}{p}} y_2^{\frac{1}{q}} + \cdots + x_n^{\frac{1}{p}} y_n^{\frac{1}{q}} \leq (x_1 + x_2 + \cdots + x_n)^{\frac{1}{p}} (y_1 + y_2 + \cdots + y_n)^{\frac{1}{q}}$$

holds for any positive numbers $x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n$.

$\boxed{4}$ Let a be a real number and consider the function

$$f(x,y) = \begin{cases} (x^2 + \sin^2 y)^a & (x,y) \neq (0,0), \\ 0 & (x,y) = (0,0) \end{cases}$$

defined on the plane $\mathbb{R}^2$.

(1) Find a necessary and sufficient condition in terms of a for f to be continuously differentiable on $\mathbb{R}^2$. Note that f is said to be continuously differentiable if f has continuous partial derivatives.

(2) Find a necessary and sufficient condition in terms of a for the integral

$$\iint_{0 < x^2 + y^2 \leq 1} f(x,y)\,dx\,dy$$

to be convergent.

$\boxed{5}$ Let $\mathbb{Z}$ denote the set of integers. Let X be a finite set with n elements, and let f be a mapping from X to $\mathbb{Z}$.

(1) Suppose that f is not a constant mapping. Prove the following inequality.

$$\sum_{x,y \in X} (f(x) - f(y))^2 \geq 2n - 2.$$

(2) If equality holds in (1), determine the set

$$\{\,|f^{-1}(\{k\})| : k \in \mathbb{Z}\,\},$$

where $|Y|$ denotes the cardinality of a finite set Y.

$\boxed{6}$

Let X be a random variable obeying the uniform distribution on the interval $[0,1]$, that is,

$$P(X \leq t) = \int_{-\infty}^{t} f_X(x)dx, \qquad f_X(x) = \begin{cases} 1 & \text{if } 0 \leq x \leq 1, \\ 0 & \text{otherwise.} \end{cases}$$

(1) Find the mean value $\mathbf{E}[X]$ and the variance $\mathbf{V}[X]$.

(2) Let Y be a random variable obeying the same distribution as X and assume that X and Y are independent. Set $Z = X + Y$ and $Z' = X - 2Y$. Find the covariance

$$\mathbf{Cov}(Z, Z') = \mathbf{E}[(Z - \mathbf{E}[Z])(Z' - \mathbf{E}[Z'])].$$

(3) Let Z be defined as in (2). Find the distribution function $F_Z(t) = P(Z \leq t)$ of Z and its probability density function $f_Z(x)$.

$\boxed{7}$ Let $\{y_n(x)\}_{n=1}^{\infty}$ be a sequence of differentiable functions on $\mathbb{R}$ satisfying the differential equation

$$y_n'(x) = n\, y_n(x)(1 - y_n(x)) \quad (x \in \mathbb{R}), \quad y_n(0) = \frac{1}{2}.$$

(1) Express $y_n(x)$ in terms of x and n.

(2) Show that for any bounded, continuous, and integrable function $\varphi(x)$ on $\mathbb{R}$, the following identities hold:

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} y_n(x)\,\varphi(x)dx = \int_{0}^{\infty} \varphi(x)dx, \quad \lim_{n\to\infty} \int_{-\infty}^{\infty} y_n'(x)\,\varphi(x)dx = \varphi(0).$$

$\boxed{8}$ Let $f(z) = \dfrac{\cos(\pi z)}{z^2 \sin(\pi z)}$. Answer the following questions concerning the function $f(z)$.

(1) Find all the poles of $f(z)$ in the complex plane and compute their residues.

(2) Find the value of complex line integral

$$I_n = \int_{C_n} f(z)dz$$

for each positive integer n. Here, C_n stands for the circle $|z| = n + \frac{1}{3}$ oriented anticlockwise.

$\boxed{9}$ Let X be a non-empty topological space satisfying the property $(*)$ below.

$(*)$ For any $x \in X$, there exist an open neigborhood U_x of x and an open set V_x of the two-dimensional Euclidean space $\mathbb{R}^2$ such that U_x is homeomorphic to V_x.

(1) For $p \in X$, set

$$\begin{aligned} X_p = &\{q : q \in X, \text{ there exists a continuous map } \gamma : [0,1] \to X \\ &\text{such that } \gamma(0) = p,\ \gamma(1) = q\}. \end{aligned}$$

Prove that X_p is an open set of X.

(2) Prove that if X is connected, then X is arcwise connected.

$\boxed{10}$ Let i be the imaginary unit, and let R be the following set of complex numbers

$$R = \{a + bi : a \text{ is an integer, } b \text{ is an even integer}\}.$$

(1) Show that R is a subring of the complex number field.

(2) Find all the units of R. (An element of R is a unit if its inverse is in R).

(3) An element $\alpha \in R$ is called irreducible if α is not a unit and α is divisible only by α or by units. Find irreducible elements in $\{17, 19, 3+4i\}$. Give reasons for their irreducibility.