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$\boxed{1}$ Let A be the following real 4×4 matrix:

$$A = \begin{pmatrix} a & 1 & 1 & 1 \\ 1 & a & 1 & 1 \\ 1 & 1 & a & 1 \\ 1 & 1 & 1 & a \end{pmatrix}.$$

Answer the following questions.

(1) Find the eigenvalues of A.

(2) Diagonalize A by an orthogonal matrix. Find also an orthogonal matrix which diagonalizes A.

$\boxed{2}$ For real numbers a and b, consider the vectors:

$$v_1 = \begin{pmatrix} a \\ b \\ 0 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 0 \\ a \\ b \end{pmatrix}, \quad v_3 = \begin{pmatrix} b \\ 0 \\ a \end{pmatrix}.$$

(1) Find a necessary and sufficient condition for the set of the vectors $\{v_1, v_2, v_3\}$ to form a basis of $\mathbb{R}^3$.

(2) Show that the mapping $f : \mathbb{R}^3 \to \mathbb{R}^3$ defined by

$$f(\begin{pmatrix} x \\ y \\ z \end{pmatrix}) = \begin{pmatrix} z \\ y \\ x \end{pmatrix}$$

is linear.

(3) Under the condition of (1), find the matrix representation of the mapping f in (2) with respect to the basis $\{v_1, v_2, v_3\}$.

$\boxed{3}$

(1) Show that $\dfrac{d}{dx}\arctan x = \dfrac{1}{1+x^2}$.

(2) Show that

$$\frac{1}{1+x^2} = \sum_{k=0}^{n}(-1)^k x^{2k} + \frac{(-1)^{n+1}x^{2n+2}}{1+x^2}.$$

(3) For every $a \in (-1,1)$, show that

$$\arctan a = \sum_{n=0}^{\infty}\frac{(-1)^n a^{2n+1}}{2n+1}.$$

$\boxed{4}$ Set $D = \{(x,y) \mid x \geq 0,\ y \geq 0\}$. For a real constant m, determine whether the integral $\displaystyle\iint_D \frac{dxdy}{(1+x^2+y^2)^m}$ is convergent or divergent. If the integral is convergent, calculate its value.

$\boxed{5}$ The cardinality of a finite set Z is denoted by $|Z|$. For a fixed finite set X and a subset A of X, the complement of A in X is denoted by A^c. The power set of X, that is, the set of all subsets of X is denoted by 2^X. Let d be the function on $2^X \times 2^X$ defined by

$$d(A,B) = \min\{|A \cup B| - |A \cap B|, |A \cup B^c| - |A \cap B^c|\}.$$

The maximum and minimum of the function d is denoted by $d_{\max}$ and $d_{\min}$, respectively.

(1) Find $d_{\min}$.

(2) Find a necessary and sufficient condition for $A, B \in 2^X$ to satisfy $d(A,B) = d_{\min}$.

(3) Find $d_{\max}$.

$\boxed{6}$ Let U_1, U_2 be independent random variables obeying the uniform distribution on $[0, 1]$ and set

$$X = \max\{U_1, U_2\}, \quad Y = \min\{U_1, U_2\}.$$

Answer the following questions.

(1) Find a function f_X (the probability density function of X) satisfying

$$P(X \le x) = \int_{-\infty}^{x} f_X(t)dt$$

for any real number x.

(2) Calculate the mean value $\mathbf{E}[X]$ and the variance $\mathbf{V}[X]$.

(3) For $0 \le x \le 1$ and $0 \le y \le 1$, find the probability $P(X \le x, Y \le y)$.

(4) Calculate the correlation coefficient of X, Y defined by

$$r = \frac{\mathbf{E}[XY] - \mathbf{E}[X]\mathbf{E}[Y]}{\sqrt{\mathbf{V}[X]\mathbf{V}[Y]}}.$$

$\boxed{7}$ Consider the following ordinary differential equation of second order:

$$(*) \qquad x''(t) = t\,x(t) \quad (t \in \mathbb{R}).$$

(1) Assuming that $x(t) = \sum_{n=0}^{\infty} a_n t^n$ satisfies (*), find a recurrence relation for the coefficients a_n.

(2) If a solution of (*) in the form of an infinite series satisfies $x(0) = 1$ and $x'(0) = 0$, show that

$$x(t) = \sum_{n=0}^{\infty} \frac{\Gamma(\frac{2}{3})t^{3n}}{9^n n!\Gamma(n + \frac{2}{3})},$$

where $\Gamma(s)$ stands for the Gamma function.

$\boxed{8}$ Answer the following questions concerning the function $f(z) = \dfrac{z^2}{e^z - 1}$.

(1) Show that the origin is a removable singularity of the function $f(z)$, and find the first five coefficients a_0, a_1, a_2, a_3, a_4 of the power series expansion $f(z) = a_0 + a_1 z + a_2 z^2 + a_3 z^3 + a_4 z^4 + \cdots$ about the origin.

(2) Find all the poles of $f(z)$ in the complex plane and compute their residues.

(3) Let C be the circle $|z| = 10$ oriented anticlockwise. Find the value of the complex line integral

$$I = \int_C f(z) dz.$$

$\boxed{9}$ Let S^2 be the 2-dimensional sphere, which is defined by

$$S^2 = \{x = (x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1^2 + x_2^2 + x_3^2 = 1\}.$$

Let $h : \mathbb{R}^2 \to \mathbb{R}$ be the smooth function defined by

$$h(y) = y_1^2 + 2y_2^2 \quad \text{for} \quad y = (y_1, y_2) \in \mathbb{R}^2.$$

(1) For the pair $\big(\frac{\partial h}{\partial y_1}(y), \frac{\partial h}{\partial y_2}(y)\big)$ of partial derivatives of h at $y = (y_1, y_2) \in \mathbb{R}^2$, find all vectors $(u_1, u_2) \in \mathbb{R}^2$ which have inner product zero, that is,

$$\Big(\frac{\partial h}{\partial y_1}(y), \frac{\partial h}{\partial y_2}(y)\Big) \cdot (u_1, u_2) = 0.$$

(2) Let $f : S^2 \to \mathbb{R}$ be a smooth function. Assume that f achieves its maximum at $p \in S^2$. Then, prove that

$$df_p(X) = 0$$

for any tangent vector $X \in T_p S^2$. Here, $df_p : T_p S^2 \to T_{f(p)}\mathbb{R}$ denotes the differential of f at $p \in S^2$.

(3) Let $\varphi : S^2 \to \mathbb{R}^2$ be a smooth map. Assume that the function $h \circ \varphi : S^2 \to \mathbb{R}$ achieves its maximum at $q \in S^2$. Prove

$$\dim\{X \in T_q S^2 \mid d\varphi_q(X) = 0\} \geq 1.$$

Here, $d\varphi_q : T_q S^2 \to T_{\varphi(q)}\mathbb{R}^2$ denotes the differential of φ at $q \in S^2$.

$\boxed{10}$ Let G be a group.

(1) If H and K are finite index subgroups of G, show that $H \cap K$ is also a finite index subgroup of G.

(2) If G has a proper subgroup of finite index, show that G has a proper normal subgroup of finite index.

(3) Consider the ring of rational integers $\mathbb{Z}$ as a group under addition. Find all the finite index subgroups of $\mathbb{Z}$.

(4) Consider the field of rational numbers $\mathbb{Q}$ as a group under addition. Find all the finite index subgroups of $\mathbb{Q}$.