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$\boxed{1}$ Let $M_2(\mathbb{R})$ be the vector space consisting of all 2×2 real matrices. For $A = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$, we define a map $L : M_2(\mathbb{R}) \to \mathbb{R}$ by

$$L(X) = \mathrm{Tr}(AX) \qquad \text{(the trace of } AX\text{)}$$

and a subset W of $M_2(\mathbb{R})$ by

$$W = \{X \in M_2(\mathbb{R}) \mid L(X) = 0\}.$$

We also define a map $T : M_2(\mathbb{R}) \to M_2(\mathbb{R})$ by

$$T(X) = {}^tX \qquad \text{(the transpose of } X\text{)}.$$

Then, answer the following questions.

(1) Prove that L is linear and that W is a linear subspace of $M_2(\mathbb{R})$.

(2) Prove that the restriction $T|_W$ of T to W is a linear map between W.

(3) For $\mathcal{B} = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & a \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & b \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & c \end{pmatrix} \right\}$, find real numbers a, b, c so that $\mathcal{B}$ is a basis of W.

(4) For a, b, c determined in (3), find the representation matrix of $T|_W$ with respect to the basis $\mathcal{B}$.

$\boxed{2}$ For real numbers a and b, set $A = \begin{pmatrix} a & b & b \\ b & a & b \\ b & b & a \end{pmatrix}$.

(1) Diagonalize A by an orthogonal matrix.

(2) Find a necessary and sufficient condition on a and b which assures that ${}^t\boldsymbol{x}A\boldsymbol{x} \geq 0$ for all $\boldsymbol{x} \in \mathbb{R}^3$.

$\boxed{3}$ Let a and b be real constants. Define a domain D in the plane by

$$D = \{(x, y) \mid x > 0,\ y > x^a\}.$$

Answer the following questions.

(1) Show that $\displaystyle\lim_{x \to +\infty} x\left(\frac{\pi}{2} - \arctan x\right) = 1$. Here $-\dfrac{\pi}{2} < \arctan x < \dfrac{\pi}{2}$.

(2) Find a necessary and sufficient condition in terms of a and b for the improper integral

$$I = \iint_D \frac{x^2 e^{-bx}}{x^2 + y^2}\,dxdy$$

to be convergent.

(3) Find the value (possibly $+\infty$) of the above I when $a = b = 1$.

$\boxed{4}$ Let $f(x)$ be a real-valued continuous function on the interval $[0, \infty)$. Assume that f is monotonically non-increasing on $[0, \infty)$ and satisfies

$$\lim_{T \to \infty} \int_0^T f(x)\ dx = +\infty.$$

Let $\{a_n\}_{n=1}^{\infty}$ be the sequence defined by

$$a_n = \sum_{k=1}^{n} f(k) \quad (n = 1, 2, \dots).$$

(1) Prove that $f(x) > 0$ for every $x \geq 0$.

(2) Prove the inequalities

$$\int_1^{n+1} f(x)\ dx \leq a_n \leq \int_0^n f(x)\ dx \quad (n = 1, 2, \dots).$$

(3) Compute the limit $\displaystyle\lim_{n \to \infty} \frac{a_n}{\int_1^n f(x)\ dx}$.

$\boxed{5}$ Let n be a positive integer, and let $F = \{0, 1, 2, 3, 4\}$. Define a map $d : F^n \times F^n \to \mathbb{Z}$ by

$$d(\boldsymbol{x}, \boldsymbol{y}) = |\{i \mid 1 \le i \le n,\ x_i \ne y_i\}|,$$

where $\boldsymbol{x} = (x_1, x_2, \ldots, x_n) \in F^n$, $\boldsymbol{y} = (y_1, y_2, \ldots, y_n) \in F^n$, and $|S|$ denotes the cardinality of a finite set S.

(1) Show that the inequality $d(\boldsymbol{x}, \boldsymbol{y}) + d(\boldsymbol{y}, \boldsymbol{z}) \ge d(\boldsymbol{x}, \boldsymbol{z})$ holds for all $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z} \in F^n$.

(2) Let $1 \le k \le n$, and let

$$\begin{aligned} \boldsymbol{x} &= (0, 0, \ldots, 0) \in F^n, \\ \boldsymbol{y} &= (\underbrace{0, 0, \ldots, 0}_{n-k}, \underbrace{1, 1, \ldots, 1}_{k}) \in F^n. \end{aligned}$$

Find $|\{\boldsymbol{z} \in F^n \mid d(\boldsymbol{y}, \boldsymbol{z}) = 1\}|$ and $|\{\boldsymbol{z} \in F^n \mid d(\boldsymbol{x}, \boldsymbol{z}) = k,\ d(\boldsymbol{y}, \boldsymbol{z}) = 1\}|$.

(3) Let $k, \boldsymbol{x}, \boldsymbol{y}$ be the same as in (2). Find the number of pairs $(\boldsymbol{z}, \boldsymbol{w}) \in F^n \times F^n$ satisfying

$$d(\boldsymbol{x}, \boldsymbol{z}) = d(\boldsymbol{x}, \boldsymbol{w}) = k,\ d(\boldsymbol{y}, \boldsymbol{z}) = d(\boldsymbol{y}, \boldsymbol{w}) = 1,\ d(\boldsymbol{z}, \boldsymbol{w}) = 2.$$

$\boxed{6}$ Choose a random point A from the unit disc $\Omega = \{(u, v) | u^2 + v^2 \le 1\}$ equipped with the uniform distribution. Express the coordinates of A by $(R\cos\Theta, R\sin\Theta)$, where $0 \le R \le 1$ and $0 \le \Theta < 2\pi$.

(1) Find a function f_R specified by

$$P(R \le x) = \int_{-\infty}^{x} f_R(t)dt$$

for all $x \in (-\infty, +\infty)$. (f_R is called the probability density function of R.)

(2) Calculate the mean value $\mathbf{E}[R]$ and the variance $\mathbf{V}[R]$.

(3) For $0 \le x \le 1$ and $0 \le y < 2\pi$, find the conditional probability $P(R \le x | \Theta \le y)$.

(4) Answer whether the random variables R and Θ are independent or not, together with a reason.

$\boxed{7}$ Let α be a positive real number, and let $y = f(x)$ be the solution of the initial value problem

$$y' = -\frac{y}{(1-x)^{\alpha}} \quad (0 < x < 1), \quad y(0) = 1.$$

(1) Find the solution $f(x)$.

(2) Compute the limit $\lim_{x \to 1-0} f'(x)$.

(3) Find a necessary and sufficient condition on α which assures that $\lim_{x \to 1-0} f^{(n)}(x) = 0$ for all nonnegative integers n. Here, $f^{(n)}(x)$ stands for the derivative of $f(x)$ of order n, and $f^{(0)}(x) = f(x)$.

$\boxed{8}$ Let n be a natural number and let a be a real number with $0 < a < 1$. Answer the following questions.

(1) Find all the poles and their residues for the meromorphic function

$$f(z) = \frac{z^n}{1 - a(z + z^{-1}) + a^2}$$

on the complex plane.

(2) Compute the integral

$$I = \int_C f(z)dz$$

for the function $f(z)$ given in the previous question. Here, C denotes the unit circle $|z| = 1$ oriented anticlockwise.

(3) Find a value of the definite integral

$$\int_0^{2\pi} \frac{\cos 5\theta}{1 - 2a\cos\theta + a^2} d\theta.$$

$\boxed{9}$ Let $\mathbb{C}^2$ be the complex 2-space equipped with the metric

$$d(z,w) = \sqrt{|z_1 - w_1|^2 + |z_2 - w_2|^2}\ , \quad z = (z_1, z_2),\ w = (w_1, w_2) \in \mathbb{C}^2.$$

For $z = (z_1, z_2),\ w = (w_1, w_2) \in \mathbb{C}^2$, define an equivalence relation $z \sim w$ by

$$(z_1, z_2) = (w_1, w_2) \quad \text{or} \quad (z_1, z_2) = (w_2, w_1),$$

and consider its quotient space $\mathbb{C}^2/\sim$. Then, answer the following questions.

(1) For the map $\varphi : \mathbb{C}^2 \longrightarrow \mathbb{C}^2,\ (z_1, z_2) \mapsto (z_1 + z_2, z_1 z_2)$, prove existence and uniqueness of a map $\widetilde{\varphi} : \mathbb{C}^2/\sim \longrightarrow \mathbb{C}^2$ satisfying $\widetilde{\varphi} \circ P = \varphi$. Here, P stands for the natural projection $P : \mathbb{C}^2 \longrightarrow \mathbb{C}^2/\sim$

(2) Prove that $\widetilde{\varphi}$ is a continuous map.

(3) Prove that $\mathbb{C}^2$ is homeomorphic to $\mathbb{C}^2/\sim$.

$\boxed{10}$ Let F_5 denote the finite field consisting of five elements. Let X be an indeterminate, and for $a, b \in F_5$, set

$$f_{a,b}(X) = X^2 + aX + b.$$

(1) Prove that for each $a \in F_5$, there exist exactly two elements $b \in F_5$ such that $f_{a,b}(X)$ is irreducible over F_5.

(2) Find all irreducible polynomials $f_{a,b}(X)$ over F_5.

(3) Express the product of all $f_{a,b}(X)$ in (2) as a sum of monomials.