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1 Consider the following symmetric matrix A and vector $\boldsymbol{e}$:

$$A = \begin{pmatrix} 0 & a & a & a \\ a & 0 & a & a \\ a & a & 0 & a \\ a & a & a & 0 \end{pmatrix}, \quad \boldsymbol{e} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

where a is a non-zero real number.

(1) Find the eigenvalues of A and their multiplicities.

(2) Express $\boldsymbol{e}$ as a sum of eigenvectors of A.

(3) For a positive integer n, find $A^n\boldsymbol{e}$.

2 For $n = 1, 2, \ldots,$ we define a function f_n on $\mathbb{R}$ by

$$f_n(x) = \begin{cases} n - n^2|x| & \text{if } -\frac{1}{n} \leq x \leq \frac{1}{n}, \\ 0 & \text{otherwise.} \end{cases}$$

(1) For $m = 1, 2, \ldots,$ prove that

$$\lim_{n\to\infty} \int_{-1}^{1} x^m f_n(x)dx = 0.$$

(2) For any polynomial $p(x)$, prove that

$$\lim_{n\to\infty} \int_{-1}^{1} p(x)f_n(x)dx = p(0).$$

(3) Let φ be a continuous function on $[-1, 1]$. Employing the fact that φ is uniformly approximated by a polynomial, prove that

$$\lim_{n\to\infty} \int_{-1}^{1} \varphi(x)f_n(x)dx = \varphi(0).$$

3 Define a surface S in $\mathbb{R}^3$ by

$$S = \{(x, y, z) \in \mathbb{R}^3 \mid 0 \le y \le 2x,\ z = 4 - x^2,\ z \ge 0\}.$$

(1) Draw the surface S.

(2) Find the surface area of S.

(3) Find the length of the curve formed by the intersection of S and another surface

$$y = \frac{4}{3}x^{3/2}.$$

4 Let n and q be integers at least two, and let N and Q be finite sets with n and q elements, respectively. Let X be the set of pairs consisting of a subset of N and a map from that subset to Q, i.e.,

$$X = \{(A, f) \mid A \subseteq N,\ f : A \to Q\}.$$

For $(A, f), (B, g) \in X$, we shall write $(A, f) \preceq (B, g)$ whenever $A \subseteq B$ and $f(x) = g(x)$ for every $x \in A$. The cardinality of a finite set S is denoted by $|S|$.

(1) Let $0 \le r \le s \le n$, and let (A, f) and (B, g) be elements of X such that $(A, f) \preceq (B, g)$, $|A| = r$, and $|B| = n$. Express

$$\big|\{(C, h) \in X \mid (A, f) \preceq (C, h) \preceq (B, g),\ |C| = s\}\big|$$

in terms of n, r, and s.

(2) Let $0 \le r \le n$, and let (A, f) be an element of X such that $|A| = r$. Express

$$\big|\{(C, h) \in X \mid (A, f) \preceq (C, h),\ |C| = n\}\big|$$

in terms of n, q, and r.

(3) Let $0 \le j \le r \le n$ and $0 \le s \le n$, and let (A, f) and (B, g) be elements of X such that $|A| = r$, $|B| = n$, and $\big|\{x \in A \mid f(x) = g(x)\}\big| = j$. Express

$$\left|\left\{ \big((C, h), (D, k)\big) \in X \times X \;\middle|\; \begin{array}{l} (C, h) \preceq (B, g),\ (C, h) \preceq (D, k), \\ (A, f) \preceq (D, k),\ |C| = s,\ |D| = n \end{array} \right\}\right|$$

in terms of n, q, r, s, and j.

5 Let X be a random variable obeying the exponential distribution with parameter $\lambda > 0$; that is,

$$P(\alpha \leq X \leq \beta) = \int_{\alpha}^{\beta} \lambda e^{-\lambda x} dx$$

for $0 \leq \alpha < \beta$.

(1) Find the mean value $\mathbf{E}[X]$ and the variance $\mathbf{V}[X]$ of X.

(2) Prove that $P(X > a + b|X > a) = P(X > b)$ for $a > 0$ and $b > 0$, where $P(A|B)$ stands for the conditional probability of A relative to B.

(3) Define a random variable Y by

$$Y = \begin{cases} [X] & \text{if } X \geq 0, \\ 0 & \text{if } X < 0, \end{cases}$$

where $[x]$ denotes the greatest integer not exceeding the real number x. Find the mean value $\mathbf{E}[Y]$.

6 For real-valued functions $x = x(t)$ and $y = y(t)$ $(t \in \mathbb{R})$, consider the following initial value problem:

$$\begin{cases} x' = \cos y, \\ y' = -\sin x, \end{cases}$$

$$\begin{cases} x(0) = a, \\ y(0) = \frac{\pi}{2} - a, \end{cases}$$

where a is a constant with $0 < a < \frac{\pi}{2}$. For every pair of solutions $x(t), y(t)$, we define a function $f = f(t)$ $(t \in \mathbb{R})$ by

$$f(t) = x(t) + y(t).$$

(1) Find a differential equation of second order satisfied by f.

(2) Find the values $f(0)$ and $f'(0)$.

(3) Find $f(t)$.

(4) Find the pair of solutions $x(t),\ y(t)$.

7 Define a meromorphic function $f(z)$ on the complex plane by

$$f(z) = \frac{1}{(1+z^2)(1+z^4)}.$$

(1) Find all the poles of $f(z)$ on the upper half-plane $\operatorname{Im} z > 0$ and their residues.

(2) Find the power series expansion of the function $f(z)$ about $z = 0$.

(3) Evaluate the following definite integral:

$$I = \int_{-\infty}^{\infty} \frac{dx}{(1+x^2)(1+x^4)}.$$

8 Let f be a continuous map from a metric space (X, d) to the n-dimensional Euclidean space $\mathbb{R}^n$. The map f is called a proper map if the inverse image $f^{-1}(K)$ is compact for any compact subset K of $\mathbb{R}^n$.

(1) Give an example of a proper map from $\mathbb{R}^2$ to $\mathbb{R}^3$.

(2) Give an example of a continuous map from $\mathbb{R}^2$ to $\mathbb{R}^3$ which is not a proper map.

(3) Let (X, d) be a metric space. Assume that the closed metric ball

$$B_r(p) = \{q \in X \mid d(p, q) \leq r\}$$

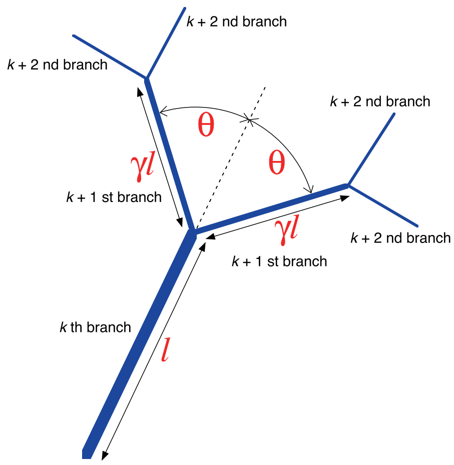
is always compact for any point p in X and any $r > 0$. Prove that (X, d) is complete.

(4) Let (X, d) be a metric space. Assume that there exists a proper map $f : (X, d) \to \mathbb{R}^n$ satisfying:

$$|f(p) - f(q)| \leq d(p, q) \quad \text{for} \quad p, q \in X,$$

where $|\cdot|$ denotes the Euclidean norm. Prove that (X, d) is complete.

9 As a simple mathematical model for repeated biological branching structure, for example, for respiratory trachea or plant vessel bifurcation, consider the following dichotomous branching in a plane:



This model satisfies the following assumptions:

- The zero-th branch is single, and its length equals l_0 ($l_0 > 0$).
- Two $(k+1)$-st branches grow from the end of a k-th branch symmetrically at angle θ ($0 < \theta \leq 90°$) with respect to the k-th branch.
- The length of every $(k+1)$-st branch is equal to that of any k-th branch multiplied by γ ($0 < \gamma \leq 1$).

In the calculation below, ignore the effect of the diameter of any branch. Two branches are said to overlap if they share a common point other than their endpoints.

(1) When $\theta = 90°$, find a condition that a fifth and another k-th branch ($k \leq 5$) overlap.

(2) When $\theta = 90°$, derive a condition that no two branches overlap.

(3) Answer just one of the following questions (A) and (B):

(A) Describe your strategy to find a condition that no two branches overlap in the case of $\theta \neq 90°$. (You do not need to derive the condition itself.)

(B) Provided that a branching structure of organism has the nature not to cause any overlap of branches, describe your opinion for the possible reason to explain such nature. (You may choose a specific structure of organism and describe your opinion according to it.)

10 Let S_3 be the symmetric group of degree 3 and let T be the set of elements in S_3 of order 2. For $a \in S_3$, let $\varphi_a : S_3 \to S_3$ denote the map defined by $\varphi_a(g) = aga^{-1}$. Let G be the automorphism group of S_3.

(1) Prove that S_3 is generated by T.

(2) For $f \in G$, prove that $f(T) = T$.

(3) Prove that $\varphi_a \in G$.

(4) Prove that the map $S_3 \to G$, $a \mapsto \varphi_a$ is an injective homomorphism.

(5) Prove that G and S_3 are isomorphic.