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1 For $n = 1, 2, \ldots$, we define a function $f_n(x)$ on $\mathbb{R}$ by

$$f_n(x) = \frac{n}{\pi(4 + n^2x^2)}.$$

(1) Find the derivative of the function $y = \arctan x$.

(2) Find the value of the definite integral $\displaystyle\int_0^{\frac{2}{n}} f_n(x)\,dx$.

(3) Find the value of the improper integral $\displaystyle\int_{-\infty}^{\infty} f_n(x)\,dx$.

(4) Let δ be a positive constant. Find the limit $\displaystyle\lim_{n\to\infty}\int_{-\delta}^{\delta} f_n(x)\,dx$.

2 Let $D = \{(x, y) \mid x + y \leq \pi,\ 0 \leq x,\ 0 \leq y\}$ and

$$f(s, t) = \iint_D \left[t\sin(x - y + s) + (x + y + t)^2\right] dxdy.$$

(1) Calculate the double integral and find $f(s, t)$.

(2) Find all the local maxima and local minima of the function $f(s, t)$ on the domain $E = \{(s, t) \mid 0 < s < 2\pi, t \in \mathbb{R}\}$.

3 Put $A = \begin{pmatrix} 1 & 1 & 1 \\ -1 & 1 & a \\ 1 & 1 & 1 \end{pmatrix}$, where a is a real number.

(1) Find the eigenvalues of A.

(2) Find the condition on a which assures that A can be diagonalized over $\mathbb{C}$.

$\boxed{4}$ Consider a cohort of n_0 members at time $t = 0$. Let us consider the death process such that the cohort becomes smaller as time passes due to the death of members. Suppose that the death of each member is independent from each other. The probability that the death for each member occurs in a sufficiently small interval $[t, t + \Delta t]$ is given by $\mu \Delta t$ where μ is a positive constant.

(1) Let $P(n, t)$ be the probability that the number of survival members in the cohort at time t is n. Derive the relation between $P(n_0, t + \Delta t)$ and $P(n_0, t)$ at first, and then the ordinary differential equation with which $P(n_0, t)$ is governed. Solve the ordinary differential equation to get a formula for $P(n_0, t)$.

(2) Express the probability that the first death occurs in $[t, t+\Delta t]$ in terms of n_0, μ, t, and Δt.

(3) Find the expected value $\langle T_1 \rangle_{n_0}$ of the time T_1 that the number of members in the cohort becomes $n_0 - 1$.

(4) Explain the relation (R) between the expected value $\langle T_e \rangle_{n_0}$ of the time T_e that the cohort of n_0 members goes extinct and $\langle T_e \rangle_{n_0 - 1}$ of T_e that so does the cohort of $n_0 - 1$ members, and find the expected value $\langle T_e \rangle_{n_0}$:

(R) $$\langle T_e \rangle_{n_0} = \langle T_1 \rangle_{n_0} + \langle T_e \rangle_{n_0 - 1}.$$

$\boxed{5}$ Let n be a positive integer and, for an integer k at least 2, let

$$P_{n,k} = \left\{ (X_1, X_2, \ldots, X_k) \;\middle|\; \begin{array}{l} X_1, X_2, \ldots, X_k \subseteq \{1, 2, \ldots, n\}, \\ X_1 \cap X_2 \cap \cdots \cap X_k = \emptyset \end{array} \right\}.$$

The cardinality of a finite set S is denoted by $|S|$.

(1) Let ℓ be an integer such that $0 \leq \ell \leq n$. Find

$$\left| \left\{ (X_1, X_2) \;\middle|\; (X_1, X_2) \in P_{n,2},\ |X_1| = \ell \right\} \right|.$$

(2) Find $|P_{n,2}|$.

(3) Let $Q_{n,3}$ be the set of $n \times 3$ matrices with entries in $\{0,1\}$ such that each row contains at least one 0, that is,

$$Q_{n,3} = \left\{ \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & a_{n3} \end{pmatrix} \;\middle|\; \begin{array}{l} a_{i1}, a_{i2}, a_{i3} \in \{0,1\} \quad (1 \le i \le n), \\ (a_{i1}, a_{i2}, a_{i3}) \ne (1,1,1) \quad (1 \le i \le n) \end{array} \right\}.$$

Give a bijection from $P_{n,3}$ to $Q_{n,3}$.

(4) Find $|P_{n,3}|$.

$\boxed{6}$ Let X and Y be independent random variables obeying the uniform distribution on $[0,1]$. Answer the following questions.

(1) Find the mean value $\mathbf{E}[X]$ and the variance $\mathbf{V}[X]$.

(2) Find the mean value $\mathbf{E}[\,|X - Y|\,]$.

(3) Let a be a real constant and set $Z = X + aY$. Find the correlation coefficient r of X and Z, where r is defined by

$$r = \frac{\mathbf{Cov}(X,Z)}{\sqrt{\mathbf{V}[X]\mathbf{V}[Z]}}, \qquad \mathbf{Cov}(X,Z) = \mathbf{E}[(X - \mathbf{E}[X])(Z - \mathbf{E}[Z])].$$

$\boxed{7}$ Define a meromorphic function $f(z)$ on the complex plane by

$$f(z) = \frac{1}{1 + z^5}.$$

(1) Find all the poles of $f(z)$ on the domain $D = \{z \in \mathbb{C} \mid \operatorname{Im} z > 0, 0 < \arg z < \frac{2\pi}{5}\}$ and their residues.

(2) For $R > 0$, let C_R be the curve given by $z = Re^{i\theta}$ $(0 \le \theta \le \frac{2\pi}{5})$. Show that

$$\lim_{R \to +\infty} \int_{C_R} f(z)dz = 0.$$

(3) Evaluate the integral

$$\int_0^{+\infty} \frac{dx}{1+x^5}.$$

$\boxed{8}$ Let $q = q(t)$ be a continuous negative function defined on $\mathbb{R}$. Let $y = y(t)$ be a real-valued solution of the following differential equation (L) on $\mathbb{R}$.

$$\text{(L)} \qquad y'' + q(t)y = 0.$$

(1) Set $z = y^2$. Show that $z'' \geq 0$ on $\mathbb{R}$.

(2) Show that if y is bounded on $\mathbb{R}$, then y must be the trivial solution.

(3) Show that if y has two zeros, then y must be the trivial solution.

$\boxed{9}$ Let $\mathbb{R}$ be the set of real numbers with the standard topology. For $x, y \in \mathbb{R}$, we denote $x \sim y$ if $x - y$ is an integer.

(1) Show that $\sim$ is an equivalence relation.

(2) Show that the quotient space $\mathbb{R}/\!\sim$ is homeomorphic to the circle

$$S^1 = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\},$$

where S^1 is equipped with the relative topology induced by $\mathbb{R}^2$.

$\boxed{10}$ Let S_3 be the symmetric group of degree 3 and let H be the subgroup of S_3 generated by all elements in S_3 of order 3.

(1) Find all elements in H.

(2) Prove that H is a normal subgroup of S_3.

(3) Prove that S_3 is not cyclic.

(4) Find all subgroups of S_3.