August, 2014

$\boxed{1}$ For a real constant α, put $f_\alpha(x) = x^\alpha \log x$ $(x > 0)$.

(1) Draw the graph of $y = f_{-2}(x)$. Include the intersection of the graph with the x-axis, the extrema, the inflection points, the behaviors of the function as $x \to 0$ and $x \to \infty$.

(2) Find a primitive function of $f_\alpha(x)$.

(3) Find a condition on α such that the improper integral $\displaystyle\int_0^1 f_\alpha(x)\,dx$ converges. Moreover, evaluate the integral under that condition.

$\boxed{2}$ Find the minimum and the maximum of the function

$$f(x, y) = \cos x + \cos y + \cos(x + y)$$

defined on $\mathbb{R}^2$, and determine all the points (x, y) at which f attains its minimum or maximum.

$\boxed{3}$ Let $M_4(\mathbb{R})$ be the vector space consisting of all 4×4 real matrices. Consider the following matrices A, B, I, and J:

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad I = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad J = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}.$$

For real numbers a, b, and c, define the matrix X by $X = a(A + {}^tA) + bI + cJ$, where ${}^tA$ denotes the transpose of A.

(1) Find a basis of $\mathbb{R}^4$ consisting of eigenvectors of the symmetric matrix $A + {}^tA$.

(2) Show that the subset

$$W = \{M \in M_4(\mathbb{R}) \mid AM = MA,\ BM = MB\}$$

of $M_4(\mathbb{R})$ is a vector subspace of $M_4(\mathbb{R})$.

(3) Show that the matrix X belongs to W.

(4) Find the matrix representing X with respect to the basis of $\mathbb{R}^4$ found in (1).

$\boxed{4}$ Consider the following mathematical model about succession and extinction of a family line originated by an ancestor. We call successors of the ancestor the first generation, those of the first generation the second, etc. The k th generation is subsequently defined. Suppose that the number of successors for each member of the family line is determined independently of each other. The probability that the number of successors equals n for a member, is now assumed to be given by $p_n = (1-a)a^n$ $(n = 0, 1, 2, \dots)$, where a is a constant satisfying $0 < a < 1$.

If the number of successors for all the k th successors is zero, we say that the family line goes extinct at the k th generation. Let d_j denote the probability that the family line goes extinct by the j th generation.

(1) Find the probability d_0 that the family line goes extinct at the ancestor (i.e., the 0 th generation).

(2) Explain the reason of the following relation:

$$(*) \qquad d_{j+1} = \sum_{n=0}^{\infty} p_n \cdot {d_j}^n \quad (j = 0, 1, 2, \dots).$$

(3) Applying $(*)$, prove that the sequence $\{d_j\}$ is strictly increasing.

(4) Find the probability that the family line eventually goes extinct, i.e., $d = \lim_{j\to\infty} d_j$.

$\boxed{5}$ Denote by $\mathbb{F}_2 = \{0, 1\}$ the finite field of order 2. Let n be a positive integer. For $\mathbf{x} = (x_1, x_2, \dots, x_n) \in \mathbb{F}_2^n$, set

$$\mathrm{wt}(\mathbf{x}) = \big|\{i \mid 1 \le i \le n,\ x_i \neq 0\}\big|,$$

where $|S|$ denotes the cardinality of a finite set S. For a vector subspace C of the vector space $\mathbb{F}_2^n$ over $\mathbb{F}_2$, set

$$C^\perp = \{\mathbf{x} \in \mathbb{F}_2^n \mid \mathbf{x} \cdot \mathbf{y} = 0 \text{ for all } \mathbf{y} \in C\},$$

where

$$\mathbf{x} \cdot \mathbf{y} = x_1 y_1 + x_2 y_2 + \cdots + x_n y_n$$

for $\mathbf{x} = (x_1, x_2, \dots, x_n), \mathbf{y} = (y_1, y_2, \dots, y_n) \in \mathbb{F}_2^n$.

(1) For $\mathbf{x}, \mathbf{y} \in \mathbb{F}_2^n$, prove that

$$\mathrm{wt}(\mathbf{x} + \mathbf{y}) = \mathrm{wt}(\mathbf{x}) + \mathrm{wt}(\mathbf{y}) - 2(\mathbf{x} * \mathbf{y}),$$

where $\mathbf{x} * \mathbf{y} = \big|\{i \mid 1 \le i \le n,\ x_i = y_i = 1\}\big|$.

(2) Let C be a vector subspace of $\mathbb{F}_2^n$ satisfying the condition

($\sharp$) $\mathrm{wt}(\mathbf{x}) \equiv 0 \pmod 4$ for all $\mathbf{x} \in C$.

Prove that $C \subset C^{\perp}$.

(3) For $n = 4$, provide an example of C with $C \subset C^{\perp}$ for which the condition ($\sharp$) is not fulfilled.

$\boxed{6}$ Let $X_1, X_2, \ldots, X_n$ be independent, identically distributed random variables obeying the uniform distribution on $[a, b]$. Set $M = \min\{X_1, X_2, \ldots, X_n\}$.

(1) Find the distribution function of M defined by $F_M(x) = P(M \leq x)$.

(2) Find the probability density function $f_M(x)$ of M.

(3) Find the mean value and variance of M.

$\boxed{7}$ In this problem, e and i stand for the base of the natural logarithm and the imaginary unit, respectively. We define a meromorphic function f on the complex plane by

$$f(z) = \frac{ze^z}{1 + e^{4z}}.$$

(1) Find all the poles and their residues of f on the parallel strip

$$\Omega = \{z \in \mathbb{C} \mid 0 < \mathrm{Im}\, z < \pi\}.$$

(2) Consider the integral

$$I(a) = \int_0^{\pi} f(a + iy)\, dy$$

for a real number a. Show that $I(a) \to 0$ as $a \to +\infty$ or $a \to -\infty$.

(3) Prove the following identity:

$$\int_0^{+\infty} \frac{\log t}{1 + t^4}\, dt = -\frac{\pi^2}{8\sqrt{2}}.$$

$\boxed{8}$ For a real-valued function $y = y(x)$ of class C^2 on the open interval $(-\frac{\pi}{2}, \frac{\pi}{2})$, consider the differential equation

$$y'' + 2(\tan x)y' - y = 0.$$

Let $y_j = y_j(x)$ $(j = 1, 2)$ be the solutions, respectively, with the initial conditions

$$y_1(0) = 1,\ y_1'(0) = 0; \quad y_2(0) = 0,\ y_2'(0) = 1.$$

Define the function $w = w(x)$ by

$$w = y_1 y_2' - y_1' y_2.$$

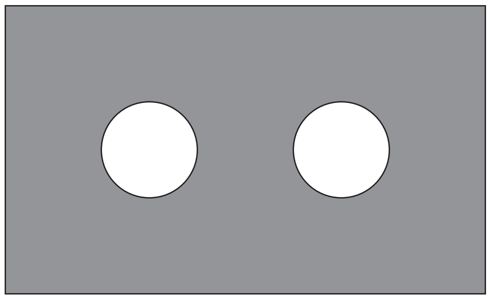
(1) Find a differential equation satisfied by w, and determine w.

(2) Show that $y_2(x) = \sin x$.

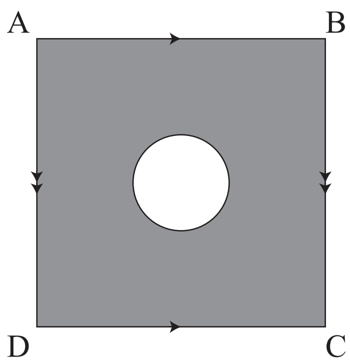
(3) Find $y_1(x)$.

(4) Solve the initial value problem: $y'' + 2(\tan x)y' - y = \cos^2 x$, $y(0) = 2$, $y'(0) = 3$.

$\boxed{9}$ Let X be a planar figure indicated by the following picture, including all the boundaries.



Let Y be a figure obtained from the following picture (including all the boundaries) by identifying AB with DC and AD with BC as indicated by the arrows.



We assume that those planar figures are equipped with the relative topology induced by the plane $\mathbb{R}^2$.

(1) Calculate the homology groups $H_i(X)$ and $H_i(Y)$ of X and Y, respectively, for $i = 0, 1, 2, \ldots$.

(2) Show that X and Y are not homeomorphic.

(3) Let I denote the closed interval $[0, 1]$. Show that $X \times I$ and $Y \times I$ are homeomorphic.

$\boxed{10}$ Let p be a prime number and let n be a positive integer. Let G be a group of order p^n, and set

$$Z(G) = \{b \in G \mid \ bc = cb \text{ for all } c \in G\}.$$

For $a \in G$, let a^G and $C_G(a)$ denote the subsets of G defined by

$$a^G = \{bab^{-1} \mid b \in G\}, \qquad C_G(a) = \{b \in G \mid ba = ab\}.$$

In the following, $|S|$ denotes the cardinality of a finite set S.

(1) Prove that $Z(G)$ and $C_G(a)$ are subgroups of G.

(2) Prove that the following two conditions (i) and (ii) are equivalent:

   (i) $|G/C_G(a)| = 1$,

   (ii) $a \in Z(G)$,

   where $G/C_G(a)$ denotes the set of all cosets of $C_G(a)$ in G.

(3) Define a map $\psi : G \to a^G$ by $\psi(b) = bab^{-1}$. Prove that ψ induces a bijection from $G/C_G(a)$ to a^G.

(4) Prove that $|Z(G)| > 1$.