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$\boxed{1}$ Let f be a twice differentiable function on an open interval I.

(1) Prove that

$$f''(x) = \lim_{h\to 0} \frac{f(x+h) - 2f(x) + f(x-h)}{h^2}$$

for all $x \in I$.

(2) Prove that $f''(x) \geq 0$ for all $x \in I$ if and only if

$$f\left(\frac{a+b}{2}\right) \leq \frac{f(a)+f(b)}{2}$$

for all $a, b \in I$.

(3) Prove that

$$\left(\frac{\sin c}{c}\right)^2 \geq \frac{\sin a}{a} \cdot \frac{\sin b}{b}$$

for all $a, b \in (0, \pi)$, with $c = \dfrac{a+b}{2}$.

$\boxed{2}$ Consider the following improper integral for a real number α :

$$F(\alpha) = \iint_D \sin^\alpha(x^2+y^2)\,dxdy,$$

where $D = \{(x,y) \in \mathbb{R}^2 \mid 0 < x^2+y^2 < \pi\}$.

(1) Characterize convergence of the improper integral $F(\alpha)$ in terms of α.

(2) Prove

$$\int_0^1 x^{a-1}(1-x)^{b-1}\,dx = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} \quad (a, b > 0),$$

where $\Gamma(s) = \displaystyle\int_0^\infty e^{-t}\,t^{s-1}\,dt$.

(3) When the improper integral $F(\alpha)$ is convergent, express $F(\alpha)$ in terms of values of the function $\Gamma(s)$.

$\boxed{3}$ Let k be a constant and put $A = \begin{pmatrix} 2 & 1 & 0 \\ 1 & k & 3 \\ 0 & 3 & 2 \end{pmatrix}$. Assume that there exists a 2×3 matrix X such that ${}^tX\,X = A$. Here ${}^tX$ denotes the transposed matrix of X.

(1) Find k and the rank of A.

(2) Find a matrix X all of whose entries are integers.

(3) Find A^n for a positive integer n.

$\boxed{4}$ Consider the following ordinary differential equation as a mathematical model for the temporal variation of a bioresource stock which is consumed in a constant rate:

$$\frac{dN(t)}{dt} = \{1 - N(t)\}\, N(t) - hN(t),$$

where $N(t)$ is the stock size of the bioresource at time t, and h is the consumption rate. Assume that h is a positive constant and $N(0) = N_0 > 0$.

(1) Determine $N(t)$.

(2) Find the condition for h that the bioresource does not go extinct.

(3) Find the value of h so as to maximize the consumed amount of the bioresource per unit time at the stationary state.

$\boxed{5}$ Let v, k, t be positive integers with $v \geq k \geq t$, and let Ω be a v-element set. Let $\mathcal{B}$ be a family of k-element subsets of Ω. Assume that for any t-element subset T of Ω, there exists a unique element $B \in \mathcal{B}$ such that $T \subset B$. The cardinality of a finite set X is denoted by $|X|$.

(1) Express $|\mathcal{B}|$ in terms of v, k and t.

(2) Let i be a positive integer at most t. For an i-element subset S of Ω, express $|\{B \in \mathcal{B} \mid S \subset B\}|$ in terms of v, k, t and i.

(3) For $v = 7$, $k = 3$, and $t = 2$, provide an example of a pair of Ω and $\mathcal{B}$.

$\boxed{6}$ Consider a game of "Head or Tail," respectively associated to 1 and 0. Let X_n be the result of the n-th trial, and assume that $\mathbb{P}(X_n = 1) = p$, where p is a constant independent of n with $0 < p < 1$. Define a random variable T by

$$T = \inf\{\, n \mid X_{n-1} \neq X_n,\ n \geq 2\,\},$$

where we put $T = \infty$ in the case that $\{\, n \mid X_{n-1} \neq X_n,\ n \geq 2\,\} = \emptyset$.

(1) Find $\mathbb{P}(A_n \cap A_{n+1})$ for $n \geq 2$, where $A_n = \{X_{n-1} \neq X_n\}$.

(2) Calculate $\mathbb{P}(T = n)$, and show that $\mathbb{P}(T < \infty) = 1$.

(3) Find $\mathbb{P}(X_T = 1,\ X_{T+1} = 0)$.

$\boxed{7}$ Let $\log z$ be the natural logarithm of z which is a single-valued holomorphic branch on Ω determined by $\log 1 = 0$, where Ω is the slit domain defined as the complex plane $\mathbb{C}$ minus the negative imaginary axis $\{iy \mid y \leq 0\}$. For a constant a with $-1 < a < 1$, let $f(z)$ be a meromorprhic function on Ω given by

$$f(z) = \frac{e^{a \log z}}{1 + z^2}.$$

(1) Find the image domain of the upper half-plane $H = \{z \in \mathbb{C} \mid \operatorname{Im} z > 0\}$ under the mapping $\log z$.

(2) Find all the residues of $f(z)$ on the upper half-plane H.

(3) For a positive number R, consider the complex integral

$$I_R = \int_{\gamma_R} f(z)dz$$

on the curve $\gamma_R(t) = Re^{it}$ $(0 \leq t \leq \pi)$. Show that

$$\lim_{R \to 0+} I_R = \lim_{R \to +\infty} I_R = 0.$$

(4) Evaluate the following definite integral:

$$\int_0^{+\infty} \frac{x^a}{1 + x^2} dx.$$

$\boxed{8}$ Let $u = u(x)$ be a real-valued function of class C^2 on the interval $I = [0, 1]$, satisfying

$$u''(x) - (4x^2 + 2)u(x) = e^{-x^2}, \quad u(0) = u(1) = 0.$$

Define a space V of real-valued functions on I by

$$V = \{v \mid v \text{ is of class } C^1 \text{ on } I \text{ and satisfies } v(0) = v(1) = 0\}.$$

Let $J : V \to \mathbb{R}$ be a functional on V given by

$$J(v) = \frac{1}{2} \int_0^1 (v'(x))^2 \, dx + \int_0^1 (2x^2 + 1)v(x)^2 \, dx + \int_0^1 e^{-x^2} v(x) \, dx.$$

(1) By setting $w(x) = u(x)e^{-x^2}$, find a differential equation satisfied by $w = w(x)$.

(2) Find $u(x)$.

(3) Show the inequality $J(u) \leq J(v)$ $(v \in V)$. Moreover, find a necessary and sufficient condition that equality holds in this inequality.

$\boxed{9}$ Let $\mathbb{R}^2$ be the two-dimensional Euclidean space. For $X \subset \mathbb{R}^2$, put $\widehat{X} = X \cup \{\infty\}$, where ∞ is a point that is not in $\mathbb{R}^2$. Let $\mathcal{O}$ be a family of sets in $\widehat{X}$ defined as follows.

- If U is an open set in X, then U is in $\mathcal{O}$. Here the topology in X is induced from that of $\mathbb{R}^2$.
- If K is a compact set in X, then $(X \setminus K) \cup \{\infty\}$ is in $\mathcal{O}$. Here we regard the empty set as a compact set.

(1) Show that $(\widehat{X}, \mathcal{O})$ is a topological space with $\mathcal{O}$ a family of open sets.

(2) Show that $(\widehat{X}, \mathcal{O})$ is compact.

(3) For $X = \mathbb{R}^2$, describe $(\widehat{X}, \mathcal{O})$.

(4) For $X = \{(x, y) \in \mathbb{R}^2 \mid x \geq 0\}$, describe $(\widehat{X}, \mathcal{O})$.

$\boxed{10}$ Let F be a field, and consider the polynomial ring $F[x]$ over F. Let θ_1 and θ_2 be distinct elements of F. The principal ideal of $F[x]$ generated by an element $f(x)$ of $F[x]$ is denoted by $(f(x))$.

(1) Show that the quotient rings $F[x]/(x-\theta_1)$ and $F[x]/(x-\theta_2)$ are isomorphic as rings to F.

(2) Show that the quotient ring $F[x]/((x-\theta_1)(x-\theta_2))$ is isomorphic as a ring to the direct sum $F[x]/(x-\theta_1) \oplus F[x]/(x-\theta_2)$.

(3) Find all the maximal ideals of $F[x]/((x-\theta_1)(x-\theta_2))$.

(4) Find all the nonzero ring homomorphisms from $F[x]/((x-\theta_1)(x-\theta_2))$ to F.