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$\boxed{1}$ Define a function f on $\mathbb{R}^2$ as follows:

$$f(x,y) = \begin{cases} (x^2+y^2)^{-y/x} & x \neq 0, \\ 1 & \text{otherwise.} \end{cases}$$

(1) Determine if $f(x,y)$ is continuous at $(0,0)$.

(2) Determine if $f(x,y)$ is continuous at $(0,1)$.

(3) Evaluate the improper integral $\displaystyle\iint_D f(x,y)\,dx\,dy$, where

$$D = \{(x,y) \in \mathbb{R}^2 \mid x^2+y^2 \geq 1,\ y \geq \sqrt{3}\,x,\ x \geq 0\}.$$

$\boxed{2}$ Let $E = \{(s,t) \in \mathbb{R}^2 \mid s > 0,\ t > 0\}$. Let $f(x,y) = \sin(\sin x \,\sin y)$, and define the real-valued function φ on E by

$$\varphi(s,t) = \int_0^s dx \int_0^t f(x,y)\,dy \qquad ((s,t) \in E).$$

(1) Find the points $(s,t) \in E$ such that $\varphi_s(s,t) = 0$. Moreover, find the positive real numbers t such that $\varphi(s,t)$ is constant with respect to s. Here, φ_s denotes the partial derivative of φ with respect to s.

(2) By changing the order of integration of the double integral

$$\int_0^s dx \int_0^t f_x(x,y)\,dy,$$

show that

$$\varphi_{ss}(s,t) = \int_0^t f_x(s,y)\,dy \qquad ((s,t) \in E).$$

(3) Find the local maxima and the local minima of φ. It is not necessary to compute the respective values of φ.

$\boxed{3}$ Let a be a positive real number, and let

$$A = \begin{pmatrix} a+2 & 0 & 0 \\ 0 & 1-a & 1+a \\ 0 & 1+a & 1-a \end{pmatrix}.$$

For vectors x, y in the vector space $\mathbb{R}^3$, their standard inner product is denoted by (x,y).

(1) Find the eigenvalues and their eigenvectors of A.

(2) Show that there exists a nonzero vector x in $\mathbb{R}^3$ such that $(x, Ax) = 0$.

(3) Show that, for any 2-dimensional subspace W of $\mathbb{R}^3$, there exists a vector x of length 1 in W such that $(x, Ax) \geq 2$.

$\boxed{4}$ Let γ and q be constants satisfying $\gamma > 0$ and $0 \leq q \leq 1$. Consider the following mathematical model about the dispersal of a rumor (a kind of information) in a population:

$$\begin{aligned} x_{k+1} &= e^{-\gamma y_k} x_k, \\ y_{k+1} &= \left(1 - e^{-\gamma y_k}\right) x_k + (1 - q) y_k, \end{aligned}$$

where x_k is the number of persons who do not know the rumor (receivers), and y_k is that of persons who know and tend to transmit it (transmitters) respectively on the kth day after the rumor appears in the population. The above model governs the daily variation of those numbers. Suppose that $x_0 > 0$ and $y_0 > 0$, and that the population size is also constant.

(1) The factor $1 - e^{-\gamma y_k}$ means the probability that a receiver becomes a transmitter after the receiver is transmitted the rumor on the kth day. The positive constant γ is the coefficient reflecting the likeliness of the production of new transmitters. Describe a possible meaning of the constant q.

(2) Consider the following function about the above mathematical model:

$$V(x, y) = x + y - \frac{q}{\gamma} \log x.$$

Show that $V(x_k, y_k)$ is a constant independent of k.

(3) Show that if $x_0 \leq q/\gamma$, the outbreak of rumor transmission, that is, the increase in the number of transmitters does not occur.

(4) When $q \neq 0$, find the condition for the outbreak of rumor transmission.

$\boxed{5}$ Let n be a natural number. The power set of the set $N = \{1, 2, \ldots, n\}$ will be denoted by 2^N.

(1) Suppose $f : 2^N \to 2^N$ is a bijection preserving inclusion, that is, for all $A, B \in 2^N$, $A \subseteq B$ implies $f(A) \subseteq f(B)$. Show that there exists a permutation σ of N such that $f(A) = \sigma(A)$ holds for all $A \in 2^N$, where $\sigma(A) = \{\sigma(a) \mid a \in A\}$.

(2) Suppose $g : 2^N \to 2^N$ is a bijection reversing inclusion, that is, for all $A, B \in 2^N$, $A \subseteq B$ implies $g(A) \supseteq g(B)$. Show that $g(A \cap B) = g(A) \cup g(B)$ holds for all $A, B \in 2^N$.

(3) Express the number of bijections from 2^N to itself which reverse inclusion, in terms of n.

$\boxed{6}$ Let g be a monotone, continuous function on $\mathbb{R}$ satisfying $0 \le g \le 1$. Set

$$m = \int_0^1 g(x)\,dx.$$

Let X be a random variable following the uniform distribution on $[0, 1]$, and consider

$$Y = g(X), \quad Z = \frac{g(X) + g(1 - X)}{2}.$$

(1) Express the expectations $\mathbf{E}(Y)$, $\mathbf{E}(Z)$ and the variances $\mathrm{V}(Y)$, $\mathrm{V}(Z)$ in terms of m and g.

(2) Show that for any $x, y \in \mathbb{R}$, the following inequality is satisfied:

$$(g(x) - g(y))(g(1 - x) - g(1 - y)) \le 0.$$

(3) Show the following inequality:

$$\int_0^1 g(x)g(1 - x)\,dx \le m^2.$$

(4) Let $\{X_1, X_2, \dots\}$ be a sequence of independent, identically distributed random variables with the uniform distribution on $[0, 1]$. Compare the estimators

$$A_n = \frac{1}{2n}\sum_{i=1}^{2n} g(X_i), \quad B_n = \frac{1}{2n}\sum_{i=1}^{n}(g(X_i) + g(1 - X_i))$$

and answer which is more efficient to calculate m.

$\boxed{7}$ Let $a > 1$ be a constant and define a meromorphic function $f(z)$ on the complex plane by

$$f(z) = \frac{z}{a - e^{-iz}}.$$

Here, i denotes the imaginary unit and e denotes the base of the natural logarithm.

(1) Find all the poles of $f(z)$ and their residues.

(2) Evaluate the following improper integral:

$$\int_0^{+\infty} \frac{dy}{a + e^y}.$$

(3) By considering the contour integral of $f(z)$ along the rectangle with four vertices at $\pm\pi, \pm\pi + it$, show the following equality:

$$\int_0^{\pi} \frac{x \sin x}{1 - 2a\cos x + a^2}\, dx = \frac{\pi}{a} \log \frac{a+1}{a}.$$

$\boxed{8}$ Let $x = x(t),\ y = y(t)\ (t \in \mathbb{R})$ be two real-valued functions satisfying

$$\begin{cases} x' = (\cos t)x - (\sin t)y, \\ y' = (\sin t)x + (\cos t)y, \end{cases} \qquad \begin{cases} x(0) = 1, \\ y(0) = 0. \end{cases}$$

Define a function $f = f(t)\ (t \in \mathbb{R})$ by

$$f(t) = \{x(t)\}^2 + \{y(t)\}^2.$$

(1) Find $f(t)$.

(2) Find $x(t)$ and $y(t)$ by introducing a real-valued function $\theta = \theta(t)$ such that $x = \sqrt{f(t)}\cos\theta,\ y = \sqrt{f(t)}\sin\theta$ and $\theta(0) = 0$.

(3) Draw the curve: $x = x(t),\ y = y(t)\ (0 \le t \le 2\pi)$ on the xy-plane.

$\boxed{9}$ Let $M(3;\mathbb{R}) = \{X = (x_{ij}) \mid x_{ij} \in \mathbb{R}\ (i, j = 1, 2, 3)\}$ be the set of all real square matrices of degree 3, which is considered as a topological space by identifying it with the 9-dimensional Euclidean space $\mathbb{R}^9$. We consider the subsets

$$GL_+(3;\mathbb{R}) = \{X \in M(3;\mathbb{R}) \mid \det X > 0\},$$
$$SL(3;\mathbb{R}) = \{X \in M(3;\mathbb{R}) \mid \det X = 1\},$$
$$O(3) = \{X \in M(3;\mathbb{R}) \mid {}^t XX = E_3\}$$

of $M(3;\mathbb{R})$ as topological spaces with respect to the relative topology of $M(3;\mathbb{R})$. Here, $\det X$ denotes the determinant of X, ${}^t X$ denotes the transpose of X, and also E_3 denotes the identity matrix.

(1) Determine whether the subsets $GL_+(3;\mathbb{R})$ and $SL(3;\mathbb{R})$ are open or closed in $M(3;\mathbb{R})$.

(2) Prove that $O(3)$ is compact.

(3) Prove that $O(3)$ is not connected.

(4) Prove that $SO(3) = O(3) \cap SL(3; \mathbb{R})$ is arcwise connected.

$\boxed{10}$ Let G be a finite group and let p be a prime factor of the order of G. Let k be a central element of G. Set $G^p = \{(x_1, x_2, \ldots, x_p) \mid x_i \in G\}$ and

$$S = \{(x_1, x_2, \ldots, x_p) \in G^p \mid x_1 x_2 \cdots x_p = k\}.$$

Let τ be the map from G^p to G^p defined by

$$\tau(x_1, x_2, \ldots, x_p) = (x_2, \ldots, x_p, x_1).$$

The cardinality of a finite set X is denoted by $|X|$.

(1) Find $|S|$.

(2) Prove $\tau(S) = S$.

(3) Prove that $\big|\{a \in S \mid \tau(a) = a\}\big| \equiv 0 \pmod{p}$.

(4) Prove that G has an element of order p.