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$\boxed{1}$

(1) Show that the inequality $\log(1-x) < -x$ holds if $0 < x < 1$.

(2) For each natural number n, we set

$$a_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \log n.$$

Show that the sequence $\{a_n\}_{n=1}^{\infty}$ is strictly monotone decreasing.

(3) Show that the sequence $\{a_n\}_{n=1}^{\infty}$ converges.

$\boxed{2}$

Consider the function

$$f(x,y) = \frac{1}{x^2(x+y+1)}$$

on $D = \{(x,y) \in \mathbb{R}^2 \mid 1 + |y| \leq x\}$.

(1) Find the Jacobian determinant for the change of variables $u = x+y, v = x-y$.

(2) For a natural number n, let $D_n = \{(x,y) \in \mathbb{R}^2 \mid 1 + |y| \leq x \leq n\}$. Find the double integral

$$\iint_{D_n} f(x,y)dxdy.$$

(3) Show that the improper integral

$$\iint_{D} f(x,y)dxdy$$

converges, and find the value.

$\boxed{3}$

Let a, b, c, d be real constants. Consider the following system of linear equations in variables x, y, z, w:

$$\begin{cases} x & +2y & -z & +w & = a \\ 2x & +4y & -2z & +2w & = b \\ -2x & -4y & +3z & & = c \\ 3x & +6y & -z & +7w & = d \end{cases}$$

(1) Find the rank of the coefficient matrix A of the system.

(2) Find the condition on a, b, c, d under which the system has at least one solution.

(3) Find the eigenvalues of A.

(4) Let λ be an eigenvalue of A which is an integer. Find a basis of the eigenspace of A corresponding to λ.

$\boxed{4}$ Consider the mathematical model of the temporal variation of population size (population density) about a predator and its prey, given by

$$\frac{dx(t)}{dt} = f(x, y)$$
$$\frac{dy(t)}{dt} = g(x, y),$$

where $x(t)$ and $y(t)$ denote respectively the prey population size and the predator one at time t. The functions $f(x, y)$ and $g(x, y)$ are assumed to be continuous and differentiable in x and y.

(1) The functions $f(x, y)$ and $g(x, y)$ necessarily satisfy the condition that $f_y(x, y) \le 0$ and $g_x(x, y) \ge 0$. Explain the reason.

(2) When the function $f(x, y)$ satisfies the condition that $f(0, y) > 0$, describe a possible biological assumption for this model.

(3) Assume that there exists a constant $x_c > 0$ satisfying the following condition:

$$\begin{cases} g(x, y) < 0 & (0 \le x < x_c,\ y > 0) \\ g(x, y) > 0 & (x > x_c,\ y > 0). \end{cases}$$

Describe a possible biological assumption for this model.

(4) Assume that there exists a constant $y_c > 0$ satisfiying the following condition:

$$g(x, y) < 0 \qquad (x > 0,\ 0 \le y < y_c).$$

Describe a possible biological assumption for this model.

$\boxed{5}$ Let n and k be positive integers such that $n \ge 2k$. Let X and Y be subsets of $\mathcal{N} = \{1, 2, \ldots, n\}$ such that $|X| = |Y| = k$, and let $i = |X \cap Y|$. The cardinality of a finite set A is denoted by $|A|$.

(1) For $0 \le j \le k$, express

$$\left| \left\{ Z \,\middle|\, Z \subset \mathcal{N},\ |Z| = k,\ |X \cap Z| = j \right\} \right|$$

in terms of n, k, and j.

(2) Assume that $0 \le i < k$. Express

$$\left|\{Z \mid Z \subset \mathcal{N},\ |Z| = k,\ |X \cap Z| = i + 1,\ |Y \cap Z| = k - 1\}\right|$$

in terms of n, k, and i.

(3) Assume that $0 \le i < k$. Express

$$\left|\{Z \mid Z \subset \mathcal{N},\ |Z| = k,\ |X \cap Z| = i,\ |Y \cap Z| = k - 1\}\right|$$

in terms of n, k, and i.

$\boxed{6}$ Let n be a natural number, and let $X_1, X_2, \ldots, X_n$ be independent identically distributed random variables obeying the uniform distribution on $[0, 1]$. Set $M_n = \max\{X_1, X_2, \ldots, X_n\}$ and $L_n = \min\{X_1, X_2, \ldots, X_n\}$.

(1) For a real number x, find $F(x) = P(M_2 \le x)$.

(2) Find the probability density function of M_2 and calculate the mean value of M_2.

(3) Find the probability density function of L_n and calculate the mean value of L_n.

$\boxed{7}$

(1) Find all the poles and the residues of the rational function

$$f(z) = \frac{z^2}{(z^2 + 4)(z^2 + 5)}.$$

(2) Let γ_R be the upper half of the circle $|z| = R$ oriented anti-clockwise, namely, the part of the circle contained in $\operatorname{Im} z \ge 0$. Show

$$\lim_{R \to +\infty} \int_{\gamma_R} f(z) dz = 0.$$

(3) By using the residue theorem, evaluate the improper integral

$$I = \int_0^{+\infty} \frac{x^2}{(x^2 + 4)(x^2 + 5)} dx.$$

$\boxed{8}$ Consider the differential equation

$$xy'' + 5y' - xy = 0$$

for a function $y = y(x)$. Let $y_1 = y_1(x)$ be a solution given by the power series

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^n$$

satisfying $y_1(0) = 1$. Let $y_2 = y_2(x)$ be another solution satisfying the initial conditions

$$y_2(1) = 0, \quad y_2'(1) = y_1(1).$$

(1) Determine y_1 and find the radius of convergence.

(2) Introduce a function $u = u(x)$ and set $y_2 = uy_1$. Express the coefficients a_0, a_1, and a_2 of the differential equation

$$a_2 u'' + a_1 u' + a_0 = 0$$

satisfied by u, in terms of x, y_1, and y_1'.

(3) Find the solution y_2.

$\boxed{9}$ Consider the 1-form ω on $D = \{(x, y) \in \mathbb{R}^2 \mid (x, y) \neq (0, 0)\}$, defined by

$$\omega = \frac{-ydx}{x^2 + y^2} + \frac{xdy}{x^2 + y^2}.$$

(1) Show that $d\omega = 0$.

(2) Let (r, θ) denote the polar coordinate. Describe ω in terms of dr and $d\theta$.

(3) Prove that there does not exist a smooth function f satisfying $\omega = df$ on D.

$\boxed{10}$ Define a mapping $f : \mathbb{Z}^3 \to \mathbb{Z}^2$ by

$$f(a, b, c) = (3a + 3c, 3b - 3c) \quad ((a, b, c) \in \mathbb{Z}^3).$$

(1) Show that f is a homomorphism of additive groups.

(2) Find a set of generators of the kernel of f.

(3) Denote by $\mathrm{Im}\, f$ the image of f. Show that the factor group $\mathbb{Z}^2/\mathrm{Im}\, f$ is finite.

(4) Show that $\mathbb{Z}^2/\mathrm{Im}\, f$ is not cyclic.