August, 2018

$\boxed{1}$ Define a function f on $\mathbb{R}^2$ as follows:

$$f(x,y) = \begin{cases} \dfrac{xy(x^2-y^2)}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0). \end{cases}$$

(1) Find the partial derivatives $f_x(x,y)$ and $f_y(x,y)$ for $(x,y) \neq (0,0)$.

(2) Find the partial derivatives $f_x(0,0)$ and $f_y(0,0)$.

(3) Find the partial derivatives $f_{xy}(0,0)$ and $f_{yx}(0,0)$.

(4) Determine if $f_{xy}(x,y)$ is continuous at $(0,0)$.

$\boxed{2}$ Put $f_s(x) = x^{-s}(\arctan x - 1)$ for a real positive number s.

(1) Evaluate the improper integral $\displaystyle\int_1^\infty f_3(x)\,dx$.

(2) Evaluate the improper integral $\displaystyle\int_1^\infty f_2(x)\,dx$.

(3) Give the condition on s that the improper integral $\displaystyle\int_1^\infty f_s(x)\,dx$ converges.

$\boxed{3}$ Let x and y be real numbers, and define the matrix A as follows:

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & x \\ 0 & -x & y \end{pmatrix}.$$

(1) Depict the range of x and y such that all the eigenvalues of A are real.

(2) Depict the range of x and y such that A has an eigenvalue with multiplicity at least two.

(3) Depict the range of x and y such that A has an eigenvalue with multiplicity at least two and is diagonalizable.

$\boxed{4}$ Consider the optimality of the dormancy of seed for an annual plant which sets seed in spring under the following conditions:

- The annual environmental fluctuation is now assumed, as the environmental condition is favorable for the reproduction with probability p, while it is unfavorable with probability $1-p$.
- The considered plant population can produce M seeds per flowering individual under the favorable environment, and m seeds per flowering individual under the unfavorable environment, where $0 \leq m < M$.
- The probability of germination in spring is given by g ($0 \leq g \leq 1$).
- The ungerminated seed becomes dormant under ground for a year, and survives till the next spring in which it germinates with probability g again.
- It is assumed that the germinated seed can necessarily grow up to the mature individual, and can produce its seeds.
- The seeds under ground is assumed to survive without any effect of the environmental fluctuation.

Let S_0 be the total number of seeds in the fall of the last year, and answer the following questions about the reproduction of this annual plant population in this year.

(1) Derive the expected total number of seeds under ground S_1 in the fall of this year.

(2) Suppose that the environment condition in the spring of this year is favorable for the reproduction while that of the next year is unfavorable. Derive the expected total number of seeds under ground S_2 in the fall of the next year. What value of g maximizes S_2?

(3) Argue the characteristics of the plant population for which S_2 is maximized by a positive g.

$\boxed{5}$ Let $\{\pm 1\}^n$ denote the set of n-dimensional vectors whose entries consist only of $1, -1$. For $\boldsymbol{a}, \boldsymbol{b} \in \{\pm 1\}^n$, let $(\boldsymbol{a}, \boldsymbol{b})$ denote their standard inner product by regarding them as vectors in $\mathbb{R}^n$.

(1) Suppose that there exist $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c} \in \{\pm 1\}^n$ such that $(\boldsymbol{a}, \boldsymbol{b}) = (\boldsymbol{b}, \boldsymbol{c}) = (\boldsymbol{c}, \boldsymbol{a}) = 0$. Show that n is divisible by 4.

(2) Suppose that there exist $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c} \in \{\pm 1\}^n$ such that $(\boldsymbol{a}, \boldsymbol{b}) = (\boldsymbol{b}, \boldsymbol{c}) = (\boldsymbol{c}, \boldsymbol{a}) = -1$. Show that $n+1$ is divisible by 4.

(3) For $n = 97, 98, 99, 100$, find respectively the largest positive integer k for which there exist $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c} \in \{\pm 1\}^n$ such that $(\boldsymbol{a}, \boldsymbol{b}) = (\boldsymbol{b}, \boldsymbol{c}) = (\boldsymbol{c}, \boldsymbol{a}) = -k$.

$\boxed{6}$ Let n be a natural number, and let $x \in [0,1]$. Let Z be a random variable obeying the binomial law $B(n,x)$. Namely, for $k = 0,1,2,\dots,n$,

$$P(Z=k) = \binom{n}{k} x^k (1-x)^{n-k}.$$

(1) Calculate the expectation $E(Z)$ and the variance $V(Z)$ when $n=1$.

(2) Calculate the expectation $E(Z)$ and the variance $V(Z)$ for any fixed natural number n.

(3) Let f be a real-valued continuous function on $[0,1]$. Show the inequality

$$\left| f(x) - E\left(f\left(\frac{Z}{n}\right)\right)\right| \leq E\left(\left| f(x) - f\left(\frac{Z}{n}\right)\right|\right)$$

for any $x \in [0,1]$.

(4) Let f be a real-valued continuous function on $[0,1]$. We define

$$B_n(f,x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right)\binom{n}{k} x^k (1-x)^{n-k}.$$

Show the following convergence:

$$\sup_{x\in[0,1]} |f(x) - B_n(f,x)| \to 0, \quad n \to \infty.$$

$\boxed{7}$ For the segment $L = \{z \in \mathbb{R} \mid -1 \leq z \leq 1\}$ on the complex plane $\mathbb{C}$, define a complex-valued function $f(z)$ on the domain $\Omega = \mathbb{C} \setminus L$ by

$$f(z) = \int_L \frac{1}{\zeta - z}\, d\zeta.$$

Here, this complex integral is over the segment L starting at -1 and terminating at 1.

(1) Show that $f(z)$ is holomorphic in Ω.

(2) Find the Taylor series expansion of $f(z)$ about $z=-2$.

(3) Find the value of the complex integral

$$\int_{|z|=7} e^z f(z)\, dz.$$

Here, the circle $|z|=7$ is oriented counterclockwise.

$\boxed{8}$ Consider the following three functions $y_j = y_j(x)$ $(j = 1, 2, 3)$ on $\mathbb{R}$ defined by

$$y_1(x) = e^{2x}\cos \pi x, \quad y_2(x) = e^{2x}\sin \pi x, \quad y_3(x) = e^x.$$

(1) Show that the functions y_1, y_2, and y_3 are linearly independent.

(2) Determine real constants a, b, and c so that all the three functions y_1, y_2, and y_3 are solutions of the differential equation $y''' + ay'' + by' + cy = 0$.

(3) Consider the differential equation determined in (2). Find its solution y satisfying the following conditions:

$$y(0) = 0, \quad y'(0) = 2, \quad y(1) = e.$$

$\boxed{9}$ Let (X, d) be a metric space. For non-empty subsets A and B of X, we define

$$\delta(A, B) = \inf \left\{ d(x, y) \,\middle|\, x \in A,\ y \in B \right\}.$$

(1) Give an example of a metric space (X, d) and non-empty closed subsets A and B such that both $A \cap B = \emptyset$ and $\delta(A, B) = 0$ are satisfied.

(2) For non-empty compact subsets A and B, prove that $\delta(A, B) = 0$ implies $A \cap B \neq \emptyset$.

$\boxed{10}$ Let S_4 be the symmetric group of degree 4, and let $\sigma = (1\ 2)(3\ 4) \in S_4$ and $\sigma' = (1\ 3)(2\ 4) \in S_4$. Set

$$N = \{\tau \in S_4 \mid \sigma\tau = \tau\sigma,\ \sigma'\tau = \tau\sigma'\}.$$

(1) Show that N is a normal subgroup of S_4.

(2) Show that the quotient group S_4/N is isomorphic to the symmetric group S_3 of degree 3.

(3) Regarding σ and σ' as elements of the symmetric group S_5 of degree 5, set

$$H = \{\tau \in S_5 \mid \sigma\tau = \tau\sigma,\ \sigma'\tau = \tau\sigma'\}.$$

Show that H is not a normal subgroup of S_5.