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1

Answer the following questions.

(1) For a C^2 function $g(x)$ defined on the real line, show that

$$\frac{d^2g}{dx^2}(x) = \lim_{h\to 0}\frac{2}{h^2}\Big(\frac{g(x+h)+g(x-h)}{2} - g(x)\Big).$$

(2) For a C^2 function $f(x,y)$ defined on the Euclidean plane, show that

$$\begin{aligned}&\frac{\partial^2 f}{\partial x^2}(x,y) + \frac{\partial^2 f}{\partial y^2}(x,y)\\ = &\lim_{h\to 0}\frac{4}{h^2}\Big(\frac{f(x+h,y)+f(x-h,y)+f(x,y+h)+f(x,y-h)}{4} - f(x,y)\Big).\end{aligned}$$

2

Consider the following matrices A and B:

$$A = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}, \qquad B = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

(1) For each of A and B, determine its eigenvalues and eigenspaces.

(2) Find all the vectors in $\mathbb{R}^3$ which are eigenvectors of both A and B.

(3) Show that the vectors found in (2) above are eigenvectors of every 3×3 matrix M such that $AM = MA$ and $BM = MB$.

(4) Show that every 3×3 matrix M such that $AM = MA$ and $BM = MB$ is diagonalizable.

3

Let $f(x)$ be a non-negative function for all $x \geq 1$ such that the improper integral $\displaystyle\int_1^\infty f(x)\,dx$ converges.

(1) For $f(x) = \dfrac{x}{x^4+1}$, find the value of the improper integral $\displaystyle\int_1^\infty f(x)\,dx$.

(2) If $f(x)$ is monotonically non-increasing, prove that $\displaystyle\lim_{x\to\infty} xf(x) = 0$.

(3) If $f(x)$ is continuous, then does $\displaystyle\lim_{x\to\infty} f(x) = 0$ hold? If it is true, prove it. If it is false, give a counter example.

4 Let $\mathcal{S}$ be the set of all 3×3 permutation matrices. Here, a permutation matrix is a square matrix that has exactly one entry of 1 in each row and each column and 0s elsewhere. A subspace of $\mathbb{C}^3$ is said to be $\mathcal{S}$-invariant if for any $P \in \mathcal{S}$ and any $w \in W$, $Pw \in W$ holds.

(1) Find all elements of $\mathcal{S}$.

(2) Find a common eigenvector of all elements of $\mathcal{S}$.

(3) Find all one-dimensional $\mathcal{S}$-invariant subspaces of $\mathbb{C}^3$.

(4) Find all $\mathcal{S}$-invariant subspaces of $\mathbb{C}^3$.

5 Consider the following ordinary differential equation as a mathematical model about the temporal variation of the frequency of users for a certain thing (e.g., a mode or a habit), say A:

$$\frac{df(t)}{dt} = \{1 - f(t)\}\, f(t) - q\{f(t)\}^{1-\gamma}$$

where $f(t)$ is the frequency of users at time t: $0 \leq f(t) \leq 1$. Parameters q and γ are positive constants, and $\gamma < 1$. Assume that $f(0) = f_0 > 0$. Answer the following questions.

(1) Explain the meaning of the term $q\{f(t)\}^{1-\gamma}$ in this mathematical model.

(2) Find the necessary and sufficient condition that a steady state different from $f(t) \equiv 0$ exists.

(3) Describe the characteristics about the temporal variation of the frequency of users for A when a steady state different from $f(t) \equiv 0$ exists.

6 Let k, n and r be positive integers. For a map $f\colon \{1, 2, \ldots, kn\} \to \{1, 2, \ldots, r\}$, we consider the following property $(*)$:

$$(*) \quad i \not\equiv j \pmod{k} \implies f(i) \neq f(j) \qquad (1 \leq i, j \leq kn).$$

(1) For $k = 2$, find the number of surjective maps $f\colon \{1, 2, \ldots, 2n\} \to \{1, 2, \ldots, r\}$ satisfying $(*)$, for each $r = 1, 2$ and 3.

(2) For $r \leq k$, find the number of surjective maps $f\colon \{1, 2, \ldots, kn\} \to \{1, 2, \ldots, r\}$ satisfying $(*)$.

(3) For $r = k + 1$, find the number of surjective maps $f\colon \{1, 2, \ldots, kn\} \to \{1, 2, \ldots, r\}$ satisfying $(*)$.

(4) For $r = k+1$, find the number of maps $f\colon \{1, 2, \ldots, kn\} \to \{1, 2, \ldots, r\}$ satisfying $(*)$.

7 Let N be a positive integer. Consider N cards numbered from 1 to N. Suppose that you draw n cards from those cards. Let T be the maximum among the numbers on the n cards you have drawn.

(1) Find the probability distribution $P(T = k)$.

(2) Find the expectation of T.

(3) Show that $\left(1 + \frac{1}{n}\right) T - 1$ is an unbiased estimator of N.

8

(1) Expand the meromorphic function

$$f(z) = \frac{1}{z^6 + 1}$$

in the complex plane into a form of power series about the origin.

(2) Let $f(z)$ be the function given above. Find all the poles and their residues of $f(z)$ in the upper half-plane $\Im z > 0$.

(3) By using the residue theorem, evaluate the definite integral

$$I = \int_0^{+\infty} \frac{dx}{x^6 + 1}.$$

9 Let $f(x)$ be a real-valued continuous function on $\mathbb{R}$. Consider the following differential equation $(*)$ for a function $y = y(x)$.

$$(*) \quad y'' + y' + f(x)y = 0.$$

Let $y_j(x)$ $(j = 1, 2)$ be the two solutions of $(*)$ satisfying

$$y_1(0) = 1,\ y_1'(0) = 0,\ y_2(0) = 0,\ y_2'(0) = 1.$$

Define the function $w = w(x)$ by

$$w = y_1 y_2' - y_2 y_1'.$$

(1) Find a differential equation satisfied by w, and determine w.

(2) Let $y = y(x)$ be the solution of the initial value problem:

$$y'' + y' + f(x)y = 2f(x),\ y(0) = 3,\ y'(0) = 5.$$

Express y in terms of y_1 and y_2.

(3) If the graphs of the two solutions $y = y_j(x)$ $(j = 1, 2)$ have two intersections $(a, y_1(a)), (b, y_1(b))$ and the inequality $y_1(x) > y_2(x)$ holds on the open interval (a, b), then show that $y_1(a) < 0 < y_1(b)$.

10 Define differential forms α, β as

$$\alpha = dx + \sin x\, dy + \cos x\, dz,$$
$$\beta = z\, dx \wedge dy + x\, dy \wedge dz - y\, dx \wedge dz.$$

For a real number $r > 0$, put

$$D = \{(x, y, z) \mid x^2 + y^2 + z^2 \leq r^2\},$$
$$E = \{(t, \varphi, \psi) \mid 0 \leq t \leq r,\ 0 \leq \varphi \leq \pi,\ 0 \leq \psi \leq 2\pi\}.$$

Moreover define $\Phi\colon E \to D$ as $\Phi(t, \varphi, \psi) = (t \sin\varphi \cos\psi, t \sin\varphi \sin\psi, t \cos\varphi)$.

(1) Evaluate $d\alpha$, $d\beta$, and $\alpha \wedge d\alpha$.

(2) Evaluate $\Phi^*(\alpha \wedge d\alpha)$.

(3) Evaluate $\displaystyle\int_D (\alpha \wedge d\alpha)$.

(4) Evaluate $\displaystyle\int_{\partial D} \beta$, where $\partial D = \{(x, y, z) \mid x^2 + y^2 + z^2 = r^2\}$.

11 Let G be a group, and let H, K be subgroups of G. The cardinality of a finite set A is denoted by $|A|$.

(1) Let L be a subgroup of G containing H. Show that $HK \cap L = H(K \cap L)$.

(2) Suppose that H and K are finite. Show that $|HK| = |H||K|/|H \cap K|$.

(3) Show by a counter example that HK is not necessarily a subgroup of G.