August, 2021

1 For a real number $\alpha > 1$, we consider the following function on $\mathbb{R}$:

$$f(x) = \begin{cases} x^{\alpha} \sin\left(\dfrac{1}{x^2}\right) & \text{if } x > 0, \\ 0 & \text{otherwise.} \end{cases}$$

(1) Prove that $f(x)$ is differentiable at $x = 0$.

(2) Find the condition on α under which $f(x)$ is of class C^1 on $\mathbb{R}$.

(3) Find the condition on α under which $f(x)$ is of class C^1 but not twice differentiable on $\mathbb{R}$.

2 Answer the following questions concerning the matrix

$$A = \begin{pmatrix} -1 & 0 & -3 \\ 4 & 1 & 6 \\ 2 & 0 & 4 \end{pmatrix}.$$

(1) Diagonalize A.

(2) Find the polynomial $f(x)$ of smallest degree such that $A^{-1} = f(A)$.

(3) Find a real number b and a polynomial $g(x)$ such that the matrix

$$B = \begin{pmatrix} 6 & 0 & 6 \\ 4b & -b & 6b \\ -4 & 0 & -4 \end{pmatrix}$$

satisfies $B = g(A)$.

3 Find the following limits. Here, log means the natural logarithm.

(1) $\displaystyle \lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{n} \cos\left(\frac{\pi}{2} \cdot \frac{k}{n}\right)$

(2) $\displaystyle \lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{n} \log\left(1 + \frac{k^3 - 1}{k^2 n}\right)$

(3) $\displaystyle \lim_{n\to\infty} \sum_{k=1}^{n} \sum_{\ell=1}^{n} \frac{1}{n^2} \left(\frac{k}{n}\right)\left(\frac{\ell}{n}\right) \sin\left(\left(\frac{k}{n}\right)\left(\frac{\ell}{n}\right)^2\right)$

4 Let W be the subspace of $\mathbb{R}^3$ spanned by the vectors $\begin{pmatrix} 2 \\ -2 \\ 2 \end{pmatrix}$, $\begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix}$, $\begin{pmatrix} 3 \\ -1 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 1 \\ 5 \\ -8 \end{pmatrix}$. Let $A = \begin{pmatrix} -4 & 3 & -5 \\ -4 & 3 & -4 \\ 1 & -1 & 2 \end{pmatrix}$ and let V be the sum of all eigenspaces of A.

(1) Find a basis of W.

(2) Find a basis of V.

(3) Find a basis and the dimension of the intersection $W \cap V$ of W and V.

(4) Prove that $W + V = \mathbb{R}^3$.

5 Consider the following mathematical model to govern the annual variation of the number of mature individuals at the beginning of breeding season at the nth year in an organism's population:

$$F_{n+1} = \sigma_{\mathrm{F}} F_n + \sigma_{\mathrm{J}} \omega m \, \mathtt{min}\big[F_n, M_n\big]$$
$$M_{n+1} = \sigma_{\mathrm{M}} M_n + \sigma_{\mathrm{J}} (1-\omega) m \, \mathtt{min}\big[F_n, M_n\big]$$

where the population makes the sexual reproduction by forming couples of mature female and male in the breeding season. F_n and M_n denote the number of mature female and male individuals respectively. Here, $\sigma_{\mathrm{F}}, \sigma_{\mathrm{M}}$ are the survival rate of mature female and male individual until the next breeding season respectively, m the mean number of newborns per reproductive couple of a mature female and a mature male, ω the female ratio in the whole newborns, σ_{J} the survival rate of immature individual to become mature by the next year, where $0 < \sigma_{\mathrm{F}}, \sigma_{\mathrm{M}}, \omega, \sigma_{\mathrm{J}} < 1$, $m > 0$. The notation $\mathtt{min}[a, b]$ means that $\mathtt{min}[a, b] = \mathtt{min}[b, a] = a$ when $a \leq b$.

(1) Describe your idea about the assumption on the formation of couples in the above model.

(2) When $F_1 = M_1$, show that the condition $F_2 = M_2$ is equivalent to

$$(*) \quad \sigma_{\mathrm{F}} + \sigma_{\mathrm{J}} \omega m = \sigma_{\mathrm{M}} + \sigma_{\mathrm{J}} (1-\omega) m$$

(3) When $F_1 = M_1$ and $F_2 = M_2$ hold, express F_n and M_n for $n > 1$ using F_1.

(4) When the condition $(*)$ is satisfied but $F_1 \neq M_1$, prove that the sex ratio of mature individuals defined by F_n and M_n asymptotically approaches $1:1$ year by year.

6 Let v, k, λ, b, r be positive integers satisfying $v > k > \lambda$. Let M be a $b \times v$ real matrix satisfying the following conditions:

(i) The entries of M are 0 or 1,

(ii) $MJ_{v\times v} = kJ_{b\times v}$,

(iii) $J_{b\times b}M = rJ_{b\times v}$,

(iv) ${}^{t}MM = (r-\lambda)E_v + \lambda J_{v\times v}$,

where E_v is the identity matrix of order v, $J_{m\times n}$ is the $m \times n$ matrix with entries 1, ${}^{t}M$ is the transpose of M.

(1) Express $\det({}^{t}MM)$ in terms of r, v, λ.

(2) Prove that $bk = vr$.

(3) Prove that $r(k-1) = \lambda(v-1)$.

(4) Prove that $b \geq v$.

$\boxed{7}$ Let X and Y be independent random variables obeying the uniform distribution on the interval $[-1, 1]$ and consider their sum $Z = X + Y$.

(1) Find the mean value $\mathbf{E}[Z]$ and the variance $\mathbf{V}[Z]$ of Z.

(2) Find the probability $P(Z \leq u)$ for a real number u.

(3) Find the probability density function of Z and depict an outline of its graph.

$\boxed{8}$ For the meromorphic function

$$f(z) = \frac{e^{iz}}{2+z^2}$$

on the complex plane, answer the following questions. Here, i denotes the imaginary unit.

(1) Find the power series expansion of $f(z)$ about $z = 0$ up to the term z^3.

(2) Find all the poles and their residues of $f(z)$ on the upper half-plane $\operatorname{Im} z > 0$.

(3) For a positive number R, let Γ_R be the upper half (that is, the intersection with the upper half-plane) of the circle $|z| = R$. Then show the following:

$$\int_{\Gamma_R} f(z)dz \to 0 \quad (R \to +\infty).$$

(4) Evaluate the following integral by the residue theorem:

$$I = \int_{-\infty}^{+\infty} \frac{\cos x}{2+x^2}dx.$$

$\boxed{9}$ Let $f = f(t)$ be a real-valued continuous function on $\mathbb{R}$ and assume that $\lim_{t\to\infty} f(t) = \alpha$ for a constant $\alpha > 0$. Now, let $x = x(t),\ y = y(t)$ be two real-valued functions of class C^1 on $\mathbb{R}$ satisfying

$$\begin{cases} x' &= f(t)x - e^{-t}y - (x^2+y^2)x, \\ y' &= e^{-t}x + f(t)y - (x^2+y^2)y, \end{cases} \qquad \begin{cases} x(0) &= 1, \\ y(0) &= 0. \end{cases}$$

Define the function $z = z(t)$ on $\mathbb{R}$ by

$$z(t) = \{x(t)\}^2 + \{y(t)\}^2.$$

(1) Find a differential equation satisfied by $z(t)$.

(2) By considering the function $z(t)^{-1}$, express $z(t)$ in terms of the function $F(t) = \int_0^t f(s)ds$.

(3) Find the limit $\lim_{t\to\infty} z(t)$.

$\boxed{10}$ For the 3-dimensional Euclidean space $\mathbb{R}^3$, we denote the standard inner product by $\langle \boldsymbol{x}, \boldsymbol{y} \rangle$ ($\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^3$) and define the length of the vector $\boldsymbol{x}$ by $|\boldsymbol{x}| = \sqrt{\langle \boldsymbol{x}, \boldsymbol{x} \rangle}$. We consider the 2-dimensional unit sphere $\mathbb{S}^2 = \{\boldsymbol{x} \in \mathbb{R}^3 \,|\, |\boldsymbol{x}| = 1\}$ as a topological space by the relative topology of $\mathbb{R}^3$. We also consider the subset

$$V = \left\{ (\boldsymbol{x}, \boldsymbol{y}) \in \mathbb{R}^3 \times \mathbb{R}^3 \,\middle|\, |\boldsymbol{x}| = |\boldsymbol{y}| = 1,\ \langle \boldsymbol{x}, \boldsymbol{y} \rangle = 0 \right\}$$

of $\mathbb{R}^6$ as a topological space by the relative topology, and define the mapping $\pi : V \longrightarrow \mathbb{S}^2$ as $\pi(\boldsymbol{x}, \boldsymbol{y}) = \boldsymbol{x}$ for $(\boldsymbol{x}, \boldsymbol{y}) \in V$.

(1) Prove that V is compact.

(2) Prove that the inverse image $\pi^{-1}(\boldsymbol{x})$ for any $\boldsymbol{x} \in \mathbb{S}^2$ is homeomorphic to the circle.

(3) Prove that V is homeomorphic to

$$SO(3) = \left\{ A : \text{a real square matrix of degree } 3 \,\middle|\, {}^tAA = E_3,\ \det A = 1 \right\}.$$

Here, ${}^tA$ denotes the transpose of A and E_3 denotes the identity matrix of degree 3, $SO(3)$ is considered as a topological space by the relative topology of $\mathbb{R}^9$.

$\boxed{11}$ Let S_5 be the symmetric group of degree 5. Let H be the subgroup of S_5 generated by $(1\ 2), (1\ 3\ 2\ 4) \in S_5$ and let Z be the center of H. Set $X = \{g^{-1}Zg \mid g \in S_5\}$, $C = \{g \in S_5 \mid gx = xg\ (\forall x \in Z)\}$ and $Y = \{g^{-1}Hg \mid g \in S_5\}$.

(1) Find the order of H.

(2) Find the order of Z.

(3) Find the number of the elements in X.

(4) Find the order of C.

(5) Find the number of the elements in Y.