August, 2022

$\boxed{1}$ Let $f(x,y) = \dfrac{e^{-e^{xy}}}{y}$.

(1) Find $\dfrac{\partial f}{\partial x}(x,y)$.

(2) Find the following integral:

$$\int_1^2 \left(\int_0^\infty \frac{\partial f}{\partial x}(x,y)\, dx \right) dy.$$

(3) Find the following integral, by changing the order of integration of the integral in (2):

$$\int_0^\infty \frac{e^{-e^x} - e^{-e^{2x}}}{x}\, dx.$$

$\boxed{2}$ Let $\alpha \in \mathbb{C}$, and set $A = \begin{pmatrix} 2 & \alpha & 2 \\ 2 & 2 & \alpha \\ \alpha & 2 & 2 \end{pmatrix}$, $\boldsymbol{b} = \begin{pmatrix} 1 \\ \alpha \\ 3 \end{pmatrix}$, $W = \{\boldsymbol{x} \in \mathbb{C}^3 \mid A\boldsymbol{x} = \boldsymbol{0}\}$.

(1) Find the condition on α under which the matrix A is nonsingular.

(2) Find all vectors $\boldsymbol{y} \in \mathbb{C}^3$ satisfying $A\boldsymbol{y} = \boldsymbol{b}$ when the matrix A is singular.

(3) Find the condition on α under which the dimension of the subspace W is at least 1.

(4) Under the condition of (3), find a basis of W.

$\boxed{3}$

(1) Suppose that a C^2 function $f(x,y)$ defined on a closed domain D in $\mathbb{R}^2$ satisfies

$$\frac{\partial^2}{\partial x^2} f(x,y) + \frac{\partial^2}{\partial y^2} f(x,y) > 0$$

on the interior of D. Show that $f(x,y)$ does not have maximum on the interior of D.

(2) Find the maximum of the function $x^2 + y^2 - 2y$ on $\{(x,y) \in \mathbb{R}^2 \mid x^2 + 4y^2 \le 1\}$.

$\boxed{4}$ Let a, b be real numbers, and assume $b \neq 0$. Let I denote the 3×3 identity matrix, and let J denote the 3×3 matrix all of whose entries are 1. Let $e_1, \ldots, e_6$ be the standard basis of $\mathbb{R}^6$. For the 6×6 matrix

$$A = \begin{pmatrix} aI & bJ \\ bJ & aI \end{pmatrix},$$

answer the following.

(1) Let W be the subspace of $\mathbb{R}^6$ generated by $v_1 = e_1 + e_2 + e_3$ and $v_2 = e_4 + e_5 + e_6$. Show that A leaves W invariant.

(2) Find the matrix representation of the linear mapping from W to W induced by A with respect to the basis v_1, v_2.

(3) Prove that A has exactly three distinct eigenvalues.

(4) Find a condition under which the matrix A is invertible, and find the inverse of A under that condition.

$\boxed{5}$ Consider the following mathematical model to describe a daily change of the frequency of individuals infected by a transmissible disease in a population:

$$\begin{aligned} S_{k+1} &= S_k e^{-\beta I_k} + q I_k; \\ I_{k+1} &= S_k (1 - e^{-\beta I_k}) + (1 - q) I_k, \end{aligned}$$

where S_k and I_k are the frequencies of non-infected and infected individuals respectively at the k th day ($k = 0, 1, 2, \ldots$). Parameters β and q satisfy that $\beta > 0$ and $0 < q < 1$. We assume that $I_0 > 0$.

Answer the following questions.

(1) Describe the meaning of parameter q.

(2) Derive the expected days for an infected individual at the k th day to recover from the disease.

(3) Derive the necessary and sufficient condition that there exists a positive value I^* such that $I_k = I^*$ for any $k \geq 0$.

$\boxed{6}$ Let n be an integer with $n \geq 2$. Let $\{\pm 1\}^n$ denote the set of n-dimensional vectors whose entries consist only of $1, -1$. For $\mathbf{x}, \mathbf{y} \in \{\pm 1\}^n$, let $\mathbf{x} \cdot \mathbf{y}$ denote their standard inner product by regarding them as vectors in $\mathbb{R}^n$. Consider the existence of n vectors $\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_n$ satisfying the following condition:

$$\begin{aligned} &\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_n \in \{\pm 1\}^n \\ &\mathbf{r}_i \cdot \mathbf{r}_j = 0 \quad (1 \leq i < j \leq n) \end{aligned} \qquad (*)$$

(1) For $n = 2$, find all pairs of two vectors $\mathbf{r}_1, \mathbf{r}_2$ satisfying the condition $(*)$.

(2) For $n = 3$, show that there do not exist three vectors $\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3$ satisfying the condition $(*)$.

(3) For $n = 2^k$ $(k = 1, 2, 3, \dots)$, show that there exist 2^k vectors $\mathbf{r}_1, \mathbf{r}_2, \dots, \mathbf{r}_{2^k}$ satisfying the condition $(*)$.

$\boxed{7}$ Let Y be a random variable with the probability density function

$$f_Y(y) = \begin{cases} 8y & 0 \le y \le \dfrac{1}{2}, \\ 0 & \text{elsewhere.} \end{cases}$$

Define the random variable U by $U = -2Y + 1$.

(1) Find the probability density function of U.

(2) Find the mean value $\mathbf{E}(U)$ and variance $\mathbf{V}(U)$ of the random variable U.

$\boxed{8}$

(1) Let D_R denote the domain defined by $0 < \arg z < \dfrac{2\pi}{5}$, $0 < |z| < R$ for a real number $R > 1$. Find all the poles and their residues of the meromorphic function

$$f(z) = \frac{1}{z^5 + 1}$$

in the domain D_R.

(2) Let Γ_R be the curve parametrized by $z = Re^{i\theta}$ $(0 \le \theta \le \dfrac{2\pi}{5})$. Show that

$$\lim_{R \to +\infty} \int_{\Gamma_R} \frac{dz}{z^5 + 1} = 0.$$

(3) By using the residue theorem, evaluate the definite integral

$$I = \int_0^{+\infty} \frac{dx}{x^5 + 1}.$$

Here, if necessary, the formula $\cos \dfrac{\pi}{5} = \dfrac{1 + \sqrt{5}}{4}$ may be used without proof.

$\boxed{9}$ Answer the following questions on the differential equation

$$\frac{dy}{dx} = -y^2 + y.$$

(1) Find all solutions.

(2) Find a solution under the condition $y(0) = a$ with $0 < a < 1$.

(3) Draw the graph of the solution of (2).

$\boxed{10}$ Let $M(2;\mathbb{R})$ be the set of all 2×2 real matrices. We regard $M(2;\mathbb{R})$ as a topological space by identifying it with the four-dimensional Euclidean space $\mathbb{R}^4$. We consider the following subspaces of $M(2;\mathbb{R})$ with relative topologies:

$$\begin{aligned} GL(2;\mathbb{R}) &= \{X \in M(2;\mathbb{R}) \mid \det X \neq 0\}, \\ SL(2;\mathbb{R}) &= \{X \in M(2;\mathbb{R}) \mid \det X = 1\}, \\ SO(2;\mathbb{R}) &= \{X \in M(2;\mathbb{R}) \mid {}^tX\, X = I\} \cap SL(2;\mathbb{R}). \end{aligned}$$

Here $\det X$ is the determinant of X, ${}^tX$ is the transpose of X, and I is the 2×2 identity matrix.

(1) Answer whether each of $GL(2;\mathbb{R})$, $SL(2;\mathbb{R})$, and $SO(2;\mathbb{R})$ is open or closed, or neither open nor closed, with reason.

(2) Answer whether each of $GL(2;\mathbb{R})$, $SL(2;\mathbb{R})$, and $SO(2;\mathbb{R})$ is compact or not, with reason.

(3) Show that $SO(2;\mathbb{R})$ is homeomorphic to the unit circle

$$S^1 = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}.$$

$\boxed{11}$ Let G be a finite group whose order is not divisible by 3. Assume that the map $\varphi : G \to G$, $a \mapsto a^3$ is a group homomorphism.

(1) Prove that φ is an isomorphism.

(2) For $x, y \in G$, prove that the following equations hold:

(a) $(xy)^2 = y^2x^2$.

(b) $[x, y]^3 = [x^3, y]$.

(c) $[x, y]^2 = [y, x^{-2}]$.

(d) $[x, y]^6 = 1$.

Here 1 is the identity element of G and $[x, y] = x^{-1}y^{-1}xy$.