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1 For $x, y \in \mathbb{R}$, set

$$f(x,y) = \begin{cases} \dfrac{e^x - e^y}{x - y} & \text{if } x \neq y, \\ e^x & \text{if } x = y. \end{cases}$$

For $a \in \mathbb{R}$, answer the following questions.

(1) Show that the function $f(x,y)$ is continuous at (a,a).

(2) Find the partial derivatives $f_x(a,a)$ and $f_y(a,a)$.

(3) Show that the function $f(x,y)$ is totally differentiable at (a,a).

2 For the matrix $A = \begin{pmatrix} 3 & -2 & 7 & -7 \\ 2 & -2 & 5 & -6 \\ 2 & 2 & 3 & 2 \\ 3 & 2 & 5 & 1 \end{pmatrix}$, consider the linear mapping $f : \mathbb{R}^4 \longrightarrow \mathbb{R}^4$ defined by $f(\boldsymbol{u}) = A\boldsymbol{u}$ $(\boldsymbol{u} \in \mathbb{R}^4)$.

(1) Find a basis of the kernel $W = \mathrm{Ker}(f)$ of the linear mapping f.

(2) Find an orthonormal basis of the orthogonal complement $W^\perp$ of W in $\mathbb{R}^4$ with respect to the standard inner product on $\mathbb{R}^4$.

(3) For the vector $\boldsymbol{v} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}$ in $\mathbb{R}^4$, find the vectors $\boldsymbol{w} \in W$ and $\boldsymbol{w}' \in W^\perp$ such that $\boldsymbol{v} = \boldsymbol{w} + \boldsymbol{w}'$.

3 Let n be a positive integer, and let

$$\varphi_n(s,t) = \begin{cases} \dfrac{s^3 t \sin(n\pi s t)}{s^2 + t^2} & (s,t) \neq (0,0), \\ 0 & (s,t) = (0,0) \end{cases}$$

for $(s,t) \in \mathbb{R}^2$.

Define the function $f_n(x,y)$ on the domain $D = \{(x,y) \in \mathbb{R}^2 \mid x > 0,\ y > 0\}$ by

$$f_n(x,y) = \iint_{[0,x]\times[0,y]} \varphi_n(s,t)\, ds dt,$$

where $[0,x] \times [0,y] = \{(s,t) \in \mathbb{R}^2 \mid 0 \le s \le x,\ 0 \le t \le y\}$.

(1) Show that $\varphi_n(s,t)$ is continuous at $(s,t)=(0,0)$.

(2) Express the partial derivatives $\dfrac{\partial f_n}{\partial x}(x,y)$ and $\dfrac{\partial f_n}{\partial y}(x,y)$ as single-variable integrals.

(3) Let $g_n(x) = f_n(x,x)$ for $x > 0$. Find $\dfrac{dg_n}{dx}(x)$.

(4) For the function $g_n(x)$ defined in (3) above, express the number of points of local extrema in the open interval $(0,1)$ in terms of n.

4

(1) Let a and b be real numbers with $b \neq 0$. Find a real number x such that the product of three matrices

$$\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & 1 \\ 1 & b \end{pmatrix} \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix}$$

is a diagonal matrix.

(2) Let a and b be real numbers with $b > 0$. Find a necessary and sufficient condition under which the matrix

$$\begin{pmatrix} a & 1 \\ 1 & b \end{pmatrix}$$

is positive definite.

(3) Let A and B be $n \times n$ real symmetric matrices, where n is a positive integer. Show that the $2n \times 2n$ real symmetric matrix

$$\begin{pmatrix} A & O \\ O & B \end{pmatrix}$$

is positive definite if and only if both A and B are positive definite, where O denotes the zero matrix.

(4) Let A and B be $n \times n$ real symmetric matrices, where n is a positive integer. Assume that B is positive definite. Show that the $2n \times 2n$ real symmetric matrix

$$\begin{pmatrix} A & I \\ I & B \end{pmatrix}$$

is positive definite if and only if $A - B^{-1}$ is positive definite, where I denotes the identity matrix.

5

For the stock size of a bioresource $N(t)$ at time t, consider the repeated consumptions of its stock size by 50% with period T:

$$\lim_{t \to kT+0} N(t) = 0.5 \left\{ \lim_{t \to kT-0} N(t) \right\} \quad (k = 0, 1, 2, \ldots).$$

Assume that the growth of the bioresource for $t \in (kT, (k+1)T)$ $(k = 0, 1, 2, \dots)$ is governed by the following ordinary differential equation, and $\lim_{t \to -0} N(t) = 1$:

$$\frac{dN(t)}{dt} = \{1 - N(t)\} N(t).$$

(1) Show $N(t)$ for $t \in (0, T)$ as a function of t.

(2) Let $N_+(k) = \lim_{t \to kT+0} N(t)$ $(k = 0, 1, 2, \dots)$. Find the relation between $N_+(k+1)$ and $N_+(k)$.

(3) Find the condition for the consumption period T in order not to exhaust the bioresource.

6 Let $\mathbb{F}_2 = \{0, 1\}$ be the field consisting of two elements, where the operations are defined by

$$0+0 = 1+1 = 0, \quad 0+1 = 1+0 = 1, \quad 0 \cdot 0 = 1 \cdot 0 = 0 \cdot 1 = 0, \quad 1 \cdot 1 = 1.$$

Let n be an integer with $n \geq 2$, and consider the n-dimensional vector space $\mathbb{F}_2^n$ over $\mathbb{F}_2$.

(1) Express the number of vectors in $\mathbb{F}_2^n$ in terms of n.

(2) For two non-zero vectors $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{F}_2^n$, show that $\boldsymbol{x}$ and $\boldsymbol{y}$ are linearly dependent over $\mathbb{F}_2$ if and only if $\boldsymbol{x} = \boldsymbol{y}$.

(3) Express the number of elements of the following set in terms of n:

$$\{(\boldsymbol{x}, \boldsymbol{y}) \mid \boldsymbol{x}, \boldsymbol{y} \in \mathbb{F}_2^n,\ \boldsymbol{x} \text{ and } \boldsymbol{y} \text{ are linearly independent over } \mathbb{F}_2\}$$

(4) Express the number of two-dimensional subspaces of $\mathbb{F}_2^n$ in terms of n.

7 Consider a linear regression model,

$$y_i = \beta_0 + \beta_1 x_i + \epsilon_i \qquad (i = 1, 2, \dots, n),$$

where ϵ_i follows a normal distribution with $\mathbf{E}(\epsilon_i) = 0$ and $\mathbf{V}(\epsilon_i) = \sigma^2$.

(1) Find the maximum likelihood estimator of the regression coefficient β_1.

(2) Verify if the maximum likelihood estimator obtained in (1) is an unbiased estimator of β_1.

8 Consider the polynomial

$$P_n(z) = 1 + z + z^2 + \cdots + z^n$$

of degree n for a positive integer n.

(1) Find all the zeros of $P_n(z)$ in the complex plane.

(2) Show that the inequality

$$|P_n(z)| \leq 2^{n+1} - 1$$

holds on the circle $|z| = 2$.

(3) Let a be a complex number with $|a| < 1$. Find the number of the solutions (counted with multiplicity) to the equation

$$P_5(z) = a$$

in the disk $|z| < 2$.

9 Consider the ordinary differential equation

$$\frac{dy}{dx} = 1 - y^2.$$

(1) Find all solutions.

(2) Find the limit $\lim_{x \to \infty} y(x)$ for $y(0) \geq -1$.

(3) Draw a sketch of the graph of the solution for $y(0) < -1$.

10 For $x, y \in \mathbb{R}$, let

$$M_{x,y} = \begin{pmatrix} y(1-x) & y^2 \\ y & x \end{pmatrix}.$$

For $k = 0, 1, 2$, consider

$$T_k = \{(x, y) \in \mathbb{R}^2 \mid \operatorname{rank} M_{x,y} = k\} \subset \mathbb{R}^2,$$

where $\operatorname{rank} M$ denotes the rank of a matrix M. Equip the two-dimensional Euclidean space $\mathbb{R}^2$ with the standard topology, and regard T_k $(k = 0, 1, 2)$ as a subspace of $\mathbb{R}^2$ with relative topology.

(1) Draw pictures of T_k for $k = 0, 1, 2$.

(2) Verify whether each of T_k $(k = 0, 1, 2)$ is open or closed, or neither open nor closed in $\mathbb{R}^2$.

(3) Verify whether each of T_k ($k = 0, 1, 2$) is connected or not.

(4) Verify whether each of T_k ($k = 0, 1, 2$) is compact or not.

11 Let n be a positive integer, and let S_n denote the symmetric group of degree n, that is, the group of all permutations on the set $\{1, 2, \ldots, n\}$. Let k be a positive integer with $1 \leq k < n$, and define subsets N and C of S_n as follows:

$$N = \{\sigma \in S_n \mid \{\sigma(1), \ldots, \sigma(k)\} = \{1, \ldots, k\}\},$$
$$C = \{\sigma \in S_n \mid \sigma(1) = 1, \ldots, \sigma(k) = k\}.$$

(1) Show that N is a subgroup of S_n.

(2) Show that C is a normal subgroup of N.

(3) Show that the factor group N/C is isomorphic to the symmetric group of degree k.

(4) Suppose $n = 2m$ is even, and let X_m be the set of all m-element subsets of $\{1, 2, \ldots, n\}$. For $A \in X_m$, denote by A^c the complement of A in $\{1, 2, \ldots, n\}$. When $m \geq 2$, show that there does not exist a permutation $\sigma \in S_n$ with the following property:

$$\text{For all } A \in X_m,\ \{\sigma(a) \mid a \in A\} = A^c.$$