A survey of stochastic optimal transportation problem

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Suppose that the probability distributions of a random particle $X(t)$ at $t = 0, 1$ are known. E. Schrödinger (1931) tried to consider $X(t)$ on $[0, 1]$ symmetrically in time. The probabilistic framework was given by S. Bernstein (1932) and the class of stochastic processes are called Bernstein/reciprocal processes. The equation that gives a Markovian Bernstein process is called Schrödinger's functional equation and were studied by R. Fortet (1940) and A. Beurling (1960), etc. B. Jamison (1975) gave the theory of stochastic differential equations for Markovian Bernstein processes. The solutions are called h-path processes with given (two) end points distributions. The variational characterization of a h-path process were given by J.C. Zambrini (1986) and P. Dai Pra (1991). T. Mikami (2004) proved that its zero-noise limit exists and solves Monge's problem with a quadratic cost and the corresponding Monge-Ampère equation. This inspired us to propose and study the problem of stochastic optimal control with given marginal distributions = stochastic optimal transportation problem (SOTP, for short).

T. Mikami and M. Thieullen (2006) and T. Mikami (2008) developped the Duality Theorem and its application. For instance, by the Duality Theorem, h-path processes can be constructed and a sufficient condition for the finiteness of the SOTP with given end points distributions was given by T. Mikami (2015). T. Mikami and M. Thieullen (2008) showed that the zero noise limit of the Duality Theorem for the SOTP with given end points distributions gives the Duality Theorem for the Monge-Kantrovich problem. T. Mikami (Kodai Math. J. 2006) proved the Duality Theorem for the Monge-Kantrovich problem by the idea of a one-step Markov control.

M. Born (1926) gave a probabilistic interpretation of the solution $\psi(t, x)$ to Schrödinger's equation: $|\psi(t, x)|^2$ is the probability density of a random particle $X(t)$. E. Nelson (1967) proposed the problem of the construction of a Markov process with given marginal distributions at all time and with a given generator. E.A. Carlen (1984, 1986) solved the problem by the idea of semigroup. W.A. Zheng (1985) did by that of stochastic calculus. See R. Carmona (1987), M. Nagasawa (1989) for different approaches. The variational characterization of a h-path process was applied, by T. Mikami (1990), for the construction of a Markov process with given marginal distributions at all time and with a given nonconstant diffusion matrix. It was completely studied by P. Cattiaux and C. Léonard (1994-1996) when the cost function is quadratic. For a more general class of cost function, T. Mikami (Appl. Math. Optim. 2006) generalized the idea of T. Mikami (1990) and gave an approach by the Duality Theorem for the SOTP with given end points distributions. The SOTP with given marginal distributions at all times was also studied systematically by T. Mikami (2008).

Because of the time limitation, we mainly discuss "our contribution" to the SOTP with given end points distributions. We refer to survey papers T. Mikami (2009) and C. Léonard (2014) for more information and references.