Local optimizers for the two-phase torsion problem in a ball

Lorenzo Cavallina

Graduate School of Information Sciences, Tohoku University

e-mail: cava@ims.is.tohoku.ac.jp

Let $\Omega \subset \mathbb{R}^N$ $(N \geq 2)$ be the open unit ball centered at the origin. Consider a subdomain $\omega \subset\subset \Omega$ with smooth interface $\partial\omega$. Moreover let σ_-, σ_+ be two positive constants and set $\sigma_\omega := \sigma_- \chi_\omega + \sigma_+ \chi_{\Omega\setminus\omega}$.

Consider the following functional:

$$E(\omega) := \int_\Omega \sigma_\omega |\nabla u_\omega|^2, \tag{1}$$

where u_ω is the solution to the following boundary value problem with piecewise constant coefficients:

$$\begin{cases} -\operatorname{div}(\sigma_\omega \nabla u_\omega) = 1 & \text{in } \Omega, \\ u_\omega = 0 & \text{on } \partial\Omega. \end{cases} \tag{P_ω}$$

Physically speaking, we think of Ω as a ball made up of two different materials while the value $E(\omega)$ corresponds to its torsional rigidity. In this talk we are concerned with finding the shape and position of ω which maximizes the functional E.

In the homogeneous case (i.e. when $\sigma_- = \sigma_+$), a famous result of Pólya states that the *ball* maximizes the torsional rigidity among the domains with fixed volume.

The two-phase optimization problem for the torsional rigidity (i.e. when $\sigma_- \neq \sigma_+$) is far more difficult to analyze, due to a larger number of the degrees of freedom.

The aim of this talk is to study the shape of the subdomains ω at which E attains extramal values. In particular we will restrict our attention to the case where ω and Ω are concentric open balls (we will take ω to be the open ball centered at the origin with radius $R \in (0, 1)$ and write $\omega := B_R$). We are going to compute the "derivative" of E with respect to suitable perturbations of ω (in a sense to be precisely defined in the talk). This technique, called *shape derivative* traces back to Hadamard and is still vastly employed nowadays for many optimization problems.

The approach we chose is pretty straightforward and, roughly speaking, consists in classifying the extremal shapes of E by determining the sign of its first and second order shape derivatives.