Solvability and convergence of solutions corresponding to a quasilinear SPDE in random environment [1]

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[1] T. Funaki, M. Hoshino, S. Sethuraman, and B. Xie, *Asymptotics of PDE in random environment by paracontrolled calculus*, to appear in Ann. Inst. Henri Poincaré Probab. Stat.
T. Funaki and B. Xie, *Global solvability and convergence to stationary solutions in singular quasilinear stochastic PDEs*, arXiv:2106.01102.

Plan of the talk

1. Aims and Motivation
2. Local solvability and continuity
3. Global solvability and convergence

Aims and Motivation

- We consider the quasilinear stochastic PDE on $\mathbb{T} = [0, 1)$ with periodic boundary condition:

$$\partial_t u = a(\nabla u)\Delta u + g(\nabla u)\xi, \tag{1.1}$$

where $\nabla = \partial_x, \Delta = \partial_x^2$, $a, g \in C^3(\mathbb{R})$, $0 < c_- \leq a(v) \leq c_+$.

- ξ is the spatial Gaussian white noise, i.e., mean 0 and covariance

$$E[\xi(x)\xi(y)] = \delta(x - y).$$

In other words, $\xi(x)$ is independent if x is different.

- It is known $\xi \in C^{-\frac{d}{2}-} := \bigcap_{\alpha < -\frac{d}{2}} C^\alpha$ on $\mathbb{T}^d$, where $C^\alpha \equiv C^\alpha(\mathbb{T}^d) = \mathcal{B}^\alpha_{\infty,\infty}(\mathbb{T}^d)$ denotes (Hölder-)Besov space with exponent $\alpha \in \mathbb{R}$.

Aims and Motivation

- In particular, ξ is a genuine generalized function.
- Roughly by Schauder estimate, we expect $u \in C^{2-\frac{d}{2}-}$ so that $a(\nabla u)$ is well-defined only when $2 - \frac{d}{2} > 1$.
- This forces us to restrict to $d = 1$.
- Two aims:
  1. Give the meaning of solution $u(t)$ of (1.1) and study the local-in-time solvability (FHXS(2020)[2]).
  2. Study the global-in-time solvability and the long time behavior of $u(t)$ of (1.1) with $a = g'$ (FX(2021)[4]).

Equation for slope ∇u

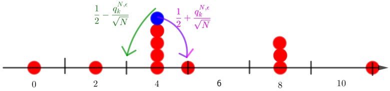
- Let φ be the primitive function of a, that is, $a = \varphi'$ and let $v := \nabla u$. Then, v solves the SPDE

$$\partial_t v = \Delta\big(\varphi(v)\big) + \nabla\big(g(v)\xi\big), \tag{1.2}$$

  since $\nabla\big(a(\nabla u)\Delta u\big) = \nabla\big(a(v)\nabla v\big) = \Delta\big(\varphi(v)\big)$. It has a mass conservation law.
- In particular, this SPDE (1.2) with $g = \varphi$ and smeared noise ξ naturally arises in a hydrodynamic scaling limit of a certain interacting particle system in a random environment.
- In case $g = \varphi$ with spatial white noise ξ, we can show the global-in-time solvability and the convergence of $v(t)$ as $t \to \infty$ to the stationary solution which is unique for each conserved quantity $\int_{\mathbb{T}} v dx$ specified by $v(0)$.

Motivation: Microscopic system

The SPDE (1.2) (but with smeared noise $\dot{w}^\epsilon$) appears in the study of the hydrodynamical behavior of a system of random walks η_t^ϵ with zero-range interactions moving in a common random environment (Landim, Pacheco, Sethuraman and Xue (2020) [7]). Let $\{q_k^{N,\epsilon}\}$ be a sequence of random variables obtained in ϵ-average.



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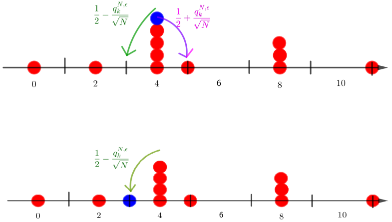

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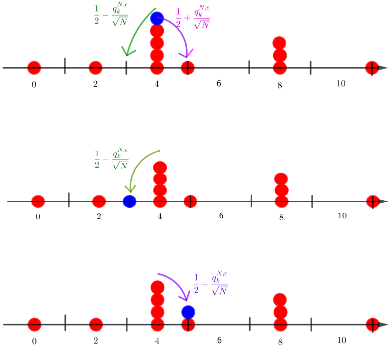

Motivation: From micro to macro

For fixed $\epsilon > 0$, as $N \to \infty$, they proved that the scaled mass empirical measure

$$\frac{1}{N}\sum_{k=1}^{N} \eta^{\epsilon}_{N^2 t}(k)\delta_{\frac{k}{N}}(dx) \longrightarrow v^{\epsilon}(t,x)dx \text{ in probability},$$

where the macroscopic density $v^{\epsilon}(t,x)$ of particles is the solution of the following PDE with mild noise:

$$\partial_t v^{\epsilon} = \Delta\{\varphi(v^{\epsilon})\} - \nabla\{\varphi(v^{\epsilon})\dot{w}^{\epsilon}(x)\}, \tag{1.3}$$

where $\varphi(v)$ is determined by the jump rate $J(k)$ and satisfies $\varphi' \geq c_- > 0$, and

$$\dot{w}^{\epsilon}(x) = \frac{1}{(a+b)\epsilon}\big(w(x+a\epsilon) - w(x-b\epsilon)\big), \ \epsilon > 0 \tag{1.4}$$

with a two-sided Brownian motion $\{w(x)\}_{x\in\mathbb{T}}$.

- The simplest example is $\varphi(v) = v$ taking $J(k) = k$.

- To realize the independent random environment, $\{q_k\}$ should be independent. However, by technical reason, LPSX [7] took smeared noise.
- It is desirable to remove this smearing procedure.
- The first aim is to consider the asymptotic behavior of (1.3) as $\epsilon \downarrow 0$.
- Instead of (1.3), we consider more general equation:

$$\partial_t v = \Delta\{\varphi(v)\} + \nabla\{g(v)\dot{w}^\epsilon(x)\}. \tag{1.5}$$

- For $\dot{w}^\epsilon(x)$ given by (1.4), we will show that $v = v^\epsilon$ converges as $\epsilon \downarrow 0$ to the solution of

$$\partial_t v = \Delta\{\varphi(v)\} + \nabla\{g(v)\dot{w}(x)\}. \tag{1.6}$$

- Noting the conservation of particle number of the underlying microscopic system, we know that (1.6) has a mass conservation law:

$$\int_{\mathbb{T}} v(t,x)dx = m\,(\in \mathbb{R}),\ {}^\forall t \geq 0.$$

Instead of (1.5) and (1.6), we actually study the equations

$$\begin{aligned}\partial_t u^\epsilon &= a(\nabla u^\epsilon)\Delta u^\epsilon + g(\nabla u^\epsilon)\cdot \xi^\epsilon,\\ \partial_t u &= a(\nabla u)\Delta u + g(\nabla u)\cdot \xi,\ x\in\mathbb{T},\end{aligned} \tag{1.7}$$

where $a(v) = \varphi'(v)$ and ξ is the spatial white noise on $\mathbb{T}$.

- If we set $v := \nabla u$, then we can recover the equation (1.6):

$$\partial_t v = \nabla\big(a(v)\nabla v\big) + \nabla\big(g(v)\cdot\xi\big) = \Delta\{\varphi(v)\} + \nabla\{g(v)\cdot\xi\}.$$

  This means that (1.7) is an integrated form of (1.6).
- Since $v = \nabla u \geq 0$, u should be $\nearrow$ in x and $u(t,1) = u(t,0) + m$. It is more natural to consider (1.7) under the modified periodic condition: $u(t, x+n) = u(t,x) + nm, \quad n\in\mathbb{Z},\ x\in\mathbb{R}$.
- However, setting $\bar{u}(t,x) := u(t,x) - mx$, we have $\bar{u}(t,x+1) = \bar{u}(t,x)$ and

$$\partial_t \bar{u} = a(\nabla\bar{u} + m)\Delta\bar{u} + g(\nabla\bar{u} + m)\cdot\xi.$$

Integrated quasilinear SPDE

- Difficulties:
  As we mentioned, (1.7) is a singular quasilinear SPDE.
  - The product $uv \in C^{\alpha\wedge\beta}$ is well-defined only $\alpha+\beta>0$.
  - But, since $\xi \in C^{-\frac{1}{2}-}$, we know that both $a(\nabla u)\Delta u$ and $g(\nabla u)\cdot\xi$ in (1.7) are ill-posed. ($\nabla u \in C^{\frac{1}{2}-}$)
  - Multiplicative noise: The classical method via a change of variable (for additive noise) can not be applied.
- The relation between (1.6) and (1.7) is similar to that of stochastic (conservative) Burgers equation and KPZ equation.

$$\partial_t u = \Delta u + \nabla u^2 + \nabla\xi(t,x), \quad x \in \mathbb{T},$$
$$\partial_t h = \Delta h + (\nabla h)^2 + \xi(t,x), \quad x \in \mathbb{T}.$$

It is known that $u = \nabla h$.

Prior researches: Local solvability

- M. Hairer made a breakthrough in solving the singular equation in 2013 and he was awarded the Fields Medal for the creation of the theory of regular structure in 2014.
- Another important method is the paracontrolled calculus originally introduced by M. Gubinelli, P. Imkeller and N. Perkowski in 2015.
- Renormalization Group, A. Kupiainen, 2015.
- Semilinear singular SPDEs are mainly studied.
- Quasilinear case: Local-in-time solvability of generalized Anderson model.

$$\partial_t u = a(u)\Delta u + g(u) \cdot \xi.$$

  - F. Otto, H. Weber (2019): Rough path based approach.
  - M. Furlan and M. Gubinelli (2019): Non-linear paracontrolled calculus.
  - I. Bailleul, A. Debussche and M. Hofmanová (2019): paracontrolled calculus.
  - M. Gerencsér, M. Hairer (2019): Regularity structure, 2019.
  - I. Bailleul, A. Mouzard (2019): High order paracontrolled calculus.

Prior researches: Global solvability and exponential conver.

- G. Cannizzaro and K. Chouk (2018): Linear equation (3.7) with $a = 1, g = v$ on $\mathbb{R}^d$.
- M. Gubinelli, P. Imkeller and N. Perkowski (2015), I. Bailleul and F. Bernicot (2016): Generalized parabolic Anderson model.
- J.-C. Mourrat and H. Weber (2017): Dynamic ϕ_3^4-model on $\mathbb{T}^3$ by establishing a priori estimate.
- M. Hoshino (2018): complex Ginzburg-Landau equation on $\mathbb{T}^3$.
- T. Funaki and M. Hoshino (2017): Multi-component coupled Kardar-Parisi-Zhang equation by studying its stationary measure.
- P. Tsatsoulis, H. Weber (2018): Exponential decay for the dynamic $P(\phi)_2$-model on $\mathbb{T}^2$ to its unique invariant measure.
- M. Gubinelli and N. Perkowski (2020): Exponential L^2-ergodicity of conservative stochastic Burgers equation on $\mathbb{T}$ based on the approach of the martingale problem.
- No result for quasilinear singular SPDEs.

Local-in-time solvability

The first main result is about the convergence and local-in-time solvability.

Theorem 2.1 (Funaki, Hoshino, Sethuraman and X. [2])

- *Assume* $a, g \in C_b^3(\mathbb{R})$ *and* $0 < c_- \leq a(v) \leq c_+$.
- *Let* $u_0 \in C^\alpha$ *with* $\alpha \in (\frac{4}{3}, \frac{3}{2})$ *be given.*
- *Let* u^ϵ *denote the solutions of the SPDE*

$$\partial_t u^\epsilon = a(\nabla u^\epsilon)\Delta u^\epsilon + g(\nabla u^\epsilon) \cdot \xi^\epsilon, \quad u^\epsilon(0) = u_0,$$

with the smeared noise $\xi^\epsilon := \psi^\epsilon * \xi$ *of* $\xi \in C^{\alpha-2}$, *where* $\psi^\epsilon(x) = \frac{1}{\epsilon}\psi(\frac{x}{\epsilon})$ *with* ψ*: measurable, compact support,* $\int_{\mathbb{R}} \psi(x)dx = 1$.

Then, there exists a random time $T > 0$ *such that as* $\epsilon \downarrow 0$,

$$u^\epsilon \to u \text{ in } \mathcal{L}_T^\alpha := C_T C^\alpha \cap C_T^{\alpha/2} L^\infty \text{ in Prob.}$$

Theorem (Continued)

and $u \in \mathcal{L}_T^\alpha$ *is a unique solution up to* T *of the SPDE* (1.7) *on* $\mathbb{T}$ *defined in paracontrolled sense. Recall that* (1.7) *is*
$\partial_t u = a(\nabla u)\Delta u + g(\nabla u) \cdot \xi, \ x \in \mathbb{T}.$
The limit u *is independent of the choice of the mollifier* ψ.

Corollary 2.2 (Comparison theorem)

Assume $a, g \in C^3([0,\infty)), g(0) = 0, \ 0 < c_- \le a(v) \le c_+$ *and* $|g'(v)| \le Ca(v)$. *Then, for the solution* $u(t)$ *of the paracontrolled SPDE* (1.7), *if* $\nabla u(0,x) \ge 0, \ \forall x \in \mathbb{T}$, *we have*

$$\nabla u(t,x) \ge 0, \ 0 \le \forall t \le T, \ x \in \mathbb{T}.$$

Proof.

If ξ is smooth, this is standard.
Then we take the limit in the noise $\hat{\xi} := (\xi, \Pi(\nabla X, \xi))$. □

Due to Funaki-Hoshino-Sethuraman-X. [2] and Funaki-X. [4], we have the continuity in initial value and enhanced noise.

Theorem 2.3 (Continuity)

Let $\alpha \in (\frac{13}{9}, \frac{3}{2})$ *and* $u(t, \hat{\xi}, u_0) \in C^\alpha$ *be the unique local paracontrolled solution of* (1.7) *with initial value* $u_0 \in C^\alpha$ *at least up to time* $T = T(\|u_0\|_{C^\alpha}, \|\hat{\xi}\|_{C^{\alpha-2}\times C^{2\alpha-2}})$.
Then, $u(t, \hat{\xi}, u_0)$ *is continuous in* $(t, \hat{\xi}, u_0)$ *in the region* $\{(t, \hat{\xi}, u_0) \in [0, \infty) \times (C^{\alpha-2} \times C^{2\alpha-3}) \times C^\alpha\,;\, t \le T\}$.

Remark 2.1

Let us consider

$$\partial_t u = a(\nabla u + m)\Delta u + g(\nabla u + m) \cdot \xi, \tag{2.1}$$

where $m \in \mathbb{R}$. *Then, we can show the solutions in continuous in* $(m, \hat{\xi}, u_0)$.

Main idea for Theorem 2.1: Paracontrolled calculus

$$
\begin{array}{ccccccc}
\xi^\epsilon & \overset{E}{\longleftrightarrow} & \Xi^\epsilon & \overset{\Phi}{\longmapsto} & U^\epsilon & \overset{\Gamma}{\longmapsto} & u^\epsilon \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\xi & \overset{E}{\longleftarrow} & \Xi & \overset{\Phi}{\longmapsto} & U & \overset{\Gamma}{\longmapsto} & u
\end{array}
$$

(Analytic part)

(Probabilistic part($\epsilon \downarrow 0$))

Enhancement $E : \xi \mapsto \Xi = (\xi, \Pi(\nabla X, \xi)) \in C^{\alpha-2} \times C^{2\alpha-3}$

Solution map $\Phi : (\eta_1, \eta_2) \mapsto U = (\eta_1, u', u^\sharp)$ (continuous map)

Projection $\Gamma : (\eta_1, u', u^\sharp) \mapsto u = \bar{\Pi}_{u'} X + u^\sharp$ (continuous map),

where $(u', u^\sharp)$ satisfies some equations. We obtain the solution of (1.7) by

$$\Gamma \circ \Phi \circ E : \xi \to u$$

Therefore, it reduces the study of (1.7) to that of the solution map Φ.
In general, when we take $\epsilon \downarrow 0$, the renormalization are usually required. In our case, the renormalization is not required.

Proof of Theorem 2.1: Paraproducts

- For two distributions u, v, due to Littlewood-Paley blocks' decomposition (based on Fourier analysis), we can define
  - $\Pi_u v (= u \prec v)$: paraproduct,
  - $\Pi(u, v) (= u \circ v)$: resonant term,
  - $\bar{\Pi}_u v (= u \prec\!\!\prec v)$: modified paraproduct (defined involving time integral).
- Littlewood-Paley decomposition of product $u \cdot v$:

  $$u \cdot v = \Pi_u v + \Pi(u, v) + \Pi_u v.$$
- Bony's estimates: Let $u \in C^\alpha, v \in C^\beta$.
  - For $\alpha > 0$ and $\beta \in \mathbb{R}$, $\|\Pi_u v\|_{C^\beta} \lesssim \|u\|_{L^\infty} \|v\|_{C^\beta}$.
  - For $\alpha \neq 0$ and $\beta \in \mathbb{R}$, $\|\Pi_u v\|_{C^{(\alpha \wedge 0)+\beta}} \lesssim \|u\|_{C^\alpha} \|v\|_{C^\beta}$.
  - For $\alpha \neq 0$, $\beta \in \mathbb{R}$ and $u \in C_T C^\alpha$, $v \in C_T C^\beta$,

    $$\|\bar{\Pi}_u v\|_{C_T C^{(\alpha \wedge 0)+\beta}} \lesssim \|u\|_{C_T C^\alpha} \|v\|_{C_T C^\beta}.$$
  - For $\alpha + \beta > 0$, $\|\Pi(u, v)\|_{C^{\alpha+\beta}} \lesssim \|u\|_{C^\alpha} \|v\|_{C^\beta}$.
  - Only for $\alpha + \beta > 0$ $(\alpha\beta \neq 0)$, we have $u \cdot v \in C^{\alpha \wedge \beta}$.

Proof of Theorem 2.1: Class of solutions

Let $\alpha \in (\frac{4}{3}, \frac{3}{2})$, $\beta \in (\frac{1}{3}, \alpha - 1) < \frac{1}{2}$, $\gamma \in (2\beta + 1, \alpha + \beta)$ be fixed.

$$\mathbf{C}_{\alpha,\beta,\gamma}(X) := \Big\{(u,u');\ \|(u,u')\|_{\alpha,\beta,\gamma} := \|u'\|_{\mathcal{L}_T^\beta} + \|u^\sharp\|_{\mathcal{L}_T^\alpha} + \sup_{0<t\le T} t^{\frac{\gamma-\alpha}{2}} \|u^\sharp(t)\|_{C^\gamma} < \infty.\Big\}$$

$$\mathcal{B}_T(\lambda) := \Big\{(u,u') \in \mathbf{C}_{\alpha,\beta,\gamma}(X);\ u(0) = u_0,\ u'(0) = \frac{g(\nabla u_0)}{a(\nabla u_0)},\ \|(u,u')\|_{\alpha,\beta,\gamma} \le \lambda\Big\}.$$

Definition 2.1 (Paracontrolled Ansatz)

We call $(u,u') \in \mathbf{C}_{\alpha,\beta,\gamma}(X)$ is paracontrolled by X if

$$u = \bar{\Pi}_{u'} X + u^\sharp, \quad u^\sharp \in \mathcal{L}_T^\alpha. \tag{2.2}$$

We have that $u^\sharp$ is a good term, while $\bar{\Pi}_{u'} X$ is a stochastic term (like stochastic integral).

Proof of Theorem 2.1: Derivation of fixed point problem

The key point is to rewrite the SPDE (1.7) as in the form

$$\mathcal{L}^0 u := \big(\partial_t - a(\nabla u_0^T)\Delta\big)u = \big(a(\nabla u) - a(\nabla u_0^T)\big)\Delta u + g(\nabla u)\cdot \xi,$$

where $u_0^T := e^{T\Delta}u_0$, the solution of $\partial_t u = \Delta u$ with $u(0) = u_0$.
By the paracontrolled calculus, we reformulate the equation (1.7) into a fixed point problem for the map Φ on $\mathcal{B}_T(\lambda)$ defined as follows:

$$\Phi(u, u') := (v, v'), \tag{2.3}$$

$$v' = \frac{g(\nabla u) - \big(a(\nabla u) - a(\nabla u_0^T)\big)u'}{a(\nabla u_0^T)}, \tag{2.4}$$

$$\begin{aligned}\mathcal{L}^0 v = {} & \Pi_{a(\nabla u_0^T)v'}\xi + g'(\nabla u)\Pi(\nabla u^\sharp, \xi) - a'(\nabla u)\Pi(\nabla u^\sharp, \bar{\Pi}_{u'}\xi) \\ & + \big(a(\nabla u) - a(\nabla u_0^T)\big)\Delta u^\sharp + \zeta.\end{aligned}$$

If (u, u') is the fixed point of Φ, then by (2.4), we have $u' = \frac{g(\nabla u)}{a(\nabla u)}$.

- The remainder term $\zeta = \zeta(u, u') \in C_T C^{\alpha+\beta-2}$ contains a quadratic function of noise $\Pi(\nabla X, \xi)$.
- Once $\Pi(\nabla X, \xi) \in C^{2\alpha-3}$ is properly defined, ζ is considered as a "good" term and also other terms are controlled by Bony's estimate and a certain commutator lemma.
- Reason to have quadratic function of noise:
  By Littlewood-Paley decomposition, we have
  $$g(\nabla u) \cdot \xi = \Pi_{g(\nabla u)}\xi + \Pi_\xi g(\nabla u) + \Pi(g(\nabla u), \xi).$$
  The paralinearization lemma (Taylor expansion) gives that
  $$\Pi(g(\nabla u), \xi) = g'(\nabla u)\Pi(\nabla u, \xi) + \text{"good"}.$$
  So, by Paracontrolled Ansatz (2.2), we have
  $$\Pi(g(\nabla u),\xi) = g'(\nabla u)u'\Pi(\nabla X,\xi) + g'(\nabla u)\Pi(\nabla u^\sharp,\xi) + \text{"good"}$$
- We reveal the ill-defined product $\Pi(\nabla X, \xi)$.
- Once it is defined (in stochastic sense), we can apply purely Fourier analytic method.

Proof of Theorem 2.1: Result for (2.3)

Theorem 2.4 (Local existence and continuity, FHSX [2], FX [4])

- *There exist a large enough $\lambda > 0$ and a small enough $T > 0$ such that the map Φ defined by* (2.3) *is contractive from $\mathcal{B}_T(\lambda)$ into itself. In particular, Φ has a unique fixed point on $[0, T]$ for $T > 0$, which solves the paracontrolled SPDE* (1.7)*, that is,*

$$\partial_t u = a(\nabla u)\Delta u + g(\nabla u) \cdot \xi, \ \ x \in \mathbb{T}$$

locally in time.

- *The map Φ depends continuously on the enhanced noise $(\hat{\xi}, u_0) \in C^{\alpha-2} \times C^{2\alpha-3} \times C^{\alpha}$, where $\hat{\xi} := (\xi, \Pi(\nabla X, \xi))$. In particular, the unique fixed point of Φ in $\mathcal{B}_T(\lambda)$ inherits the continuity in $(\hat{\xi}, u_0)$.*

Lemma 2.5 (No renormalization, FHSX[2])

Set

$$\nabla X \diamond \xi = \int_0^\infty \Pi(\nabla P_t \xi, \xi) dt.$$

Then, we have

$$E[\|\nabla X \diamond \xi\|^p_{C^{2\alpha-3}}] < \infty, \ \alpha < \frac{3}{2}.$$

Moreover, set

$$\nabla X^\epsilon = \int_0^\infty \nabla P_t \xi^\epsilon dt \left(= \nabla(-\Delta)^{-1}\xi^\epsilon \right).$$

$$\lim_{\epsilon \downarrow 0} E[\|\nabla X \diamond \xi - \Pi(\nabla X^\epsilon, \xi^\epsilon)\|^p_{C^{2\alpha-3}}] = 0,$$
$$E[\nabla X^\epsilon(x)\xi^\epsilon(x)] = E[\Pi(\nabla X^\epsilon, \xi^\epsilon)(x)] = 0.$$

So, in particular, we know that no renormalization is required.

Global solvability and convergence to stationary solution

- Let $\varphi \in C^4(\mathbb{R})$ satisfying

$$c_- \leq \varphi'(v) \leq c_+ \tag{3.1}$$

- Consider

$$\partial_t v = \Delta\{\varphi(v)\} + \nabla\{\varphi(v)\xi\} = \nabla\{\nabla\varphi(v) + \varphi(v)\xi\} \text{ on } \mathbb{T}. \tag{3.2}$$

- For a given ξ, we define its integral $\eta(x) := \langle \xi, 1_{[0,x]} \rangle \equiv \int_0^x \xi(y)dy, x \in \mathbb{T}$ and $\sigma \equiv \sigma_\xi = \eta(1)$.

$$\begin{aligned} \theta(x) \equiv \theta_\xi(x) &:= e^{-\eta(x)}\Big\{\mu \int_0^x e^{\eta(y)}dy + 1\Big\}, \quad x \in \mathbb{T}, \\ \mu \equiv \mu_\xi &:= \frac{e^{\eta(1)} - 1}{\int_0^1 e^{\eta(y)}dy}. \end{aligned} \tag{3.3}$$

- θ is continuous, periodic, and uniformly positive on $\mathbb{T}$ and satisfies

$$\nabla\theta + \xi\theta = \mu.$$

- For each conserved mass $m = \int_{\mathbb{T}} v_0(x)dx \in \mathbb{R}$, determine $z = z_m \in \mathbb{R}$ uniquely by the relation

$$m = \int_{\mathbb{T}} \varphi^{-1}(z\theta(x))dx. \tag{3.4}$$

- Then,

$$\bar{v}(x) \equiv \bar{v}_m(x) := \varphi^{-1}(z_m\theta(x)) \tag{3.5}$$

is a stationary solution of (3.2) satisfying $\int_{\mathbb{T}} vdx = m$ in distributional sense, or at least if $\xi \in C(\mathbb{T})$.

Then, (3.2) has a global solution and at least if $|\mu_\xi|$ is small enough, $\bar{v}_m$ is the unique stationary solution of the SPDE (3.2) for each fixed m.

Theorem 3.1 (Funaki-X. (2021) [4])

Let $\varphi \in C^4(\mathbb{R})$ satisfy (3.1) and $\alpha \in (\frac{13}{9}, \frac{3}{2})$. Then, for every initial value $v_0 \in C^{\alpha-1}$, the SPDE (3.2) has a global-in-time solution $v(t) \in C^{\alpha-1}$ for all $t \geq 0$.

Moreover, if $|\mu_\xi|$ is sufficiently small, $v(t)$ converges exponentially fast to $\bar{v}_m$ in $C^{\alpha-1}$ as $t \to \infty$:

$$\|v(t) - \bar{v}_m\|_{C^{\alpha-1}} \leq Ce^{-ct}, \tag{3.6}$$

for some $c, C > 0$, where m is determined from v_0 as $m = \int_{\mathbb{T}} v_0(x)dx$ and $\bar{v}_m$ is defined by (3.5).

Consider the integrated SPDE

$$\partial_t u = a(\nabla u)\Delta u + g(\nabla u)\cdot\xi,\ x\in\mathbb{T}. \tag{3.7}$$

Theorem 3.2 (Funaki-X. (2021) [4])

Assume $a = g' \in C^3(\mathbb{R})$, $0 < c_- \le\le a \le c_+$ *and* $\alpha \in (\frac{13}{9}, \frac{3}{2})$. *Then, the SPDE* (3.7) *has a global-in-time solution* $u(t)\in C^\alpha$ *for all* $t \ge 0$.
Moreover, if $|\mu_\xi|$ *is sufficiently small as in Theorem 3.1,* $u(t)$ *has the following uniform bound in* t*:*

$$\sup_{t\ge 0}\|u(t) - z_0\mu_\xi t\|_{C^\alpha} < \infty, \tag{3.8}$$

where z_0 *is defined by* (3.4) *with* $m = 0$. *In particular, we have*

$$\lim_{t\to\infty}\frac{1}{t}u(t,x) = z_0\mu_\xi$$

uniformly in $x\in\mathbb{T}$.

Proof of Theorem 3.1 (Outline)

Proof of Theorem 3.1 (Outline).

- Energy estimate: If the noise w is smooth and if the initial value $v(0)$ is nice, under a proper change of variable (nonlinear), we can find an energy functional Φ of the solution $v(t)$.
- We have Poincaré inequality for Φ, which shows the exponential convergence of $v(t)$.
- We use the continuity in enhanced noise $\hat{\xi}$, initial value and the coefficients to remove the smoothness of w, due to the Poincaré constant can be taken uniformly in an approximating sequence of the noise.
- To remove the smoothness of the initial value, we show the initial layer property, i.e., in an arbitrary short time, the solution has the desired regularity.

□

Proof of Theorem 3.1 (Details)

We assume $\xi \in C^\infty(\mathbb{T})$ and $v(t,x)$ is a smooth global-in-time solution of (3.2). We define $f(t,x)$ as

$$f(t,x) := \frac{\varphi(v(t,x))}{\theta(x)}. \tag{3.9}$$

Then, (3.2) can be rewritten as

$$\partial_t v = \nabla(\theta \nabla f + \mu f). \tag{3.10}$$

We define the functional $\Phi(f) \equiv \Phi_\theta(f)$ of $f \in H^1_\theta$ as

$$\Phi(f) \equiv \Phi_\theta(f) := \frac{1}{2}\int_{\mathbb{T}} (\nabla f)^2 \theta dx.$$

Set $K(x,f) = \varphi'(\varphi^{-1}(f\theta))\,(= \varphi'(v))$. Then $0 < c_- \le K(x,f) \le c_+$ and (3.10) can be further rewritten as

$$\partial_t f = K(x,f)(-D\Phi(x,f) + \mu\theta^{-1}\nabla f),$$

which is sometimes called Onsager equation at least when $\mu = 0$.

Let $\mathcal{D}$ be the class of all functions $v \in C^{\alpha-1}$ satisfying $\varphi(v)\theta^{-1} \in H^1$.

Proposition 3.3 (For smooth noise and good initial value)

Assume $\xi \in C^\infty(\mathbb{T})$ and $v(t,x)$ is a smooth global-in-time solution of (3.2). Then, if $f(0) \in H^1_\theta$ (equivalently $v(0) \in \mathcal{D}$), we have

$$\Phi(f(t)) \leq \Phi(f(0))e^{C(\theta)t}, \tag{3.11}$$

where $\theta = \theta_\xi$, $C(\theta) = -\frac{c_-}{2c_2(\theta)} + \frac{1}{2c_-}\mu^2 c_1(\theta)^2$, $c_1(\theta) = c_1(\min\theta)$ and

$$c_2(\theta) := \frac{1}{2}\int_{\mathbb{T}} \theta^{-1}(x)dx \int_{\mathbb{T}} \theta(y)dy, \tag{3.12}$$

the constant appearing in Poincaré inequality. In particular, if $|\mu| = |\mu_\xi|$ is small enough, $C(\theta) < 0$ and for some $c_* > 0$,

$$\Phi(f(t)) \leq \Phi(f(0))e^{-c_* t}. \tag{3.13}$$

We extend the result of Proposition 3.3 to general noise $\xi \in C^{\alpha-2}$.

Proposition 3.4 (For $\xi \in C^{\alpha-2}$ and good initial value)

Assume $v(0) \in \mathcal{D}$. Then, the solution $v(t)$ of (3.2) exists globally in time for all $t \geq 0$ and $f(t)$ defined from $v(t)$ by (3.9) satisfies

$$\Phi(f(t)) \leq e^{Ct}\Phi(f(0)), C \in \mathbb{R}. \tag{3.14}$$

In particular, if $|\mu_\xi|$ is small enough, one can take $C < 0$.

- Approximation together with Proposition 3.3.
- The constant $C(\theta)$ in (3.11) can be taken uniformly in an approximating sequence of the noise.
- We also use Theorem 2.3, that is, the continuity of the solutions in the enhanced noise, initial values and coefficients of the equation.

Corollary 3.5 (Exponential convergence)

Assume $v(0) \in \mathcal{D}$. *If* $|\mu_\xi|$ *is small enough, then*

$$\|f(t) - z_m\|_{H^1} \le Ce^{-c_* t/2}\|f(0) - z_m\|_{H^1},$$
$$\|v(t) - \bar{v}_m\|_{C^{\alpha-1}} \le Ce^{-c_* t/2}\|f(0) - z_m\|_{H^1}.$$

Lemma 3.6 (Initial layer property)

For every initial value $v(0) \in C^{\alpha-1}$ *and all* $t \in (0, T_*)$, *the solution* $v(t) \in C^{\alpha-1}$ *of the SPDE* (3.2) *in paracontrolled sense satisfies*

$$f(t) := \varphi(v(t))\theta^{-1} \in H^1, \quad \text{that is, } v(t) \in \mathcal{D}.$$

In other words, even if $f(0) \notin H^1$, *immediately after, we have* $f(t) \in H^1,\ t > 0$ *and this proves* $T_* = \infty$ *by Proposition 3.4.*

Proof of Theorem 3.2

Lemma 3.7

Assume $\xi \in C^\infty(\mathbb{T})$ and let the initial value $u_0 \in C^\alpha, \alpha \in (\frac{4}{3}, \frac{3}{2})$ of (3.7) be given. Determine $v(t)$ by solving (1.6) starting from $v_0 := \nabla u_0$, and set

$$u(t,x) := \int_0^x v(t,y)dy + \int_{\mathbb{T}} u_0(y)dy - \int_{\mathbb{T}} (1-y)v(t,y)dy \tag{3.15}$$

$$+ \int_0^t ds \int_{\mathbb{T}} g(v(s,y))\xi(y)dy =: A_1(t,x) + A_2 - A_3(t) + A_4(t)$$

Then, $u(t)$ solves the SPDE (3.7) with $a = \varphi'$.

Proof of Theorem 3.2 .

- The product $g(v(s))\xi$ is ill-posed in a classical sense, since $\chi(v(s)) \in C^{\alpha-1}$ and $\xi \in C^{\alpha-2}$ and $(\alpha - 1) + (\alpha - 2) < 0$.

□

Proof.

- However, when $a = g' = \varphi'$, $g(v(s))\xi$ is well-defined. Indeed, we have $\varphi(v(t)) = f(t)\theta$ with $f(t) \in H^1$ for $t > 0$ by Lemma 3.6 and we can show $\theta\xi \in C^{\alpha-2}$; note that in general, the product $\theta\xi$ is ill-posed. So, we have $\int_{\mathbb{T}} \chi(v(t,y))\xi(y)dy = {}_{H^1}\langle f(t), \theta\xi\rangle_{H^{-1}}, t > 0$ and then

$$u(t,x) = A_1(t,x) + A_2 - A_3(t) + \int_0^t {}_{H^1}\langle f(s), \theta\xi\rangle_{H^{-1}} ds.$$

- Then

$$\sup_{t\geq 0} \left| \int_0^t {}_{H^1}\langle f(s), \theta\xi\rangle_{H^{-1}} ds - z_0\mu t \right| < \infty.$$

and by Theorem 3.1, we obtain $\sup_{t\geq 0} |u(t) - z_0\mu t| < \infty$

□

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Thank you for your kind attention!