From random dynamics to fractional PDEs with several boundary conditions

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TÉCNICO LISBOA

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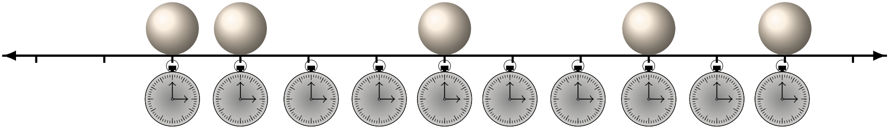
Particle system:

- N: scaling parameter;
- **Space**:
  - microscopic (discrete);
  - macroscopic (continuous);
- **Time**:
  - microscopic $t\theta(N)$;
  - macroscopic t;

- Independent Poissonian clocks;
- Transition probability $p(\cdot)$;
- $\eta_t^N(x) =$ number of particles at site x; 
- **Markov processes**; (continuous time)
- Density $\sum_x \eta_t^N(x)$ is conserved.

Exclusion: After one ring of a clock a particle jumps from x to y at rate $p(y-x)$.



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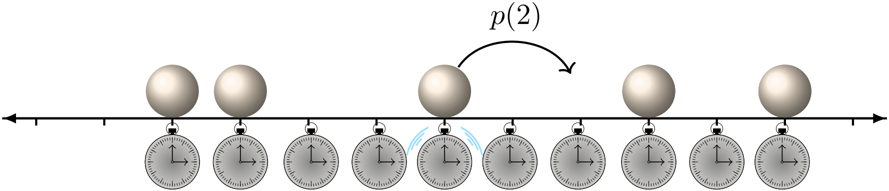

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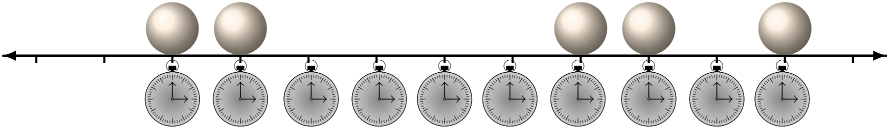

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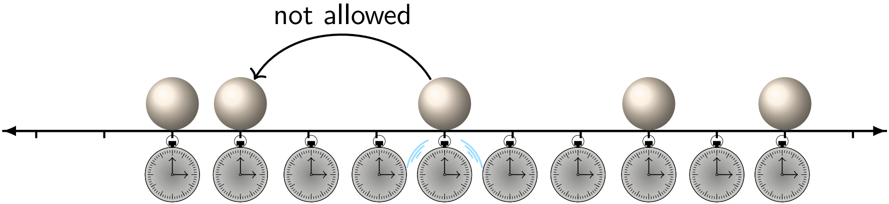

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This cannot happen.

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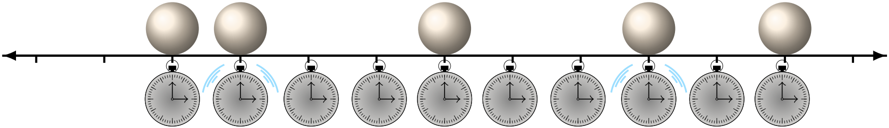

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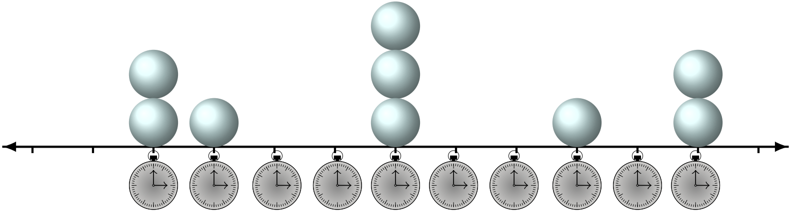
Zero-Range: after one ring of a clock one particle jumps from x to y at rate $g(\eta(x))p(y-x)$.



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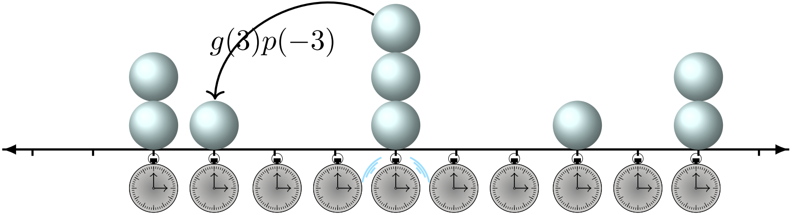

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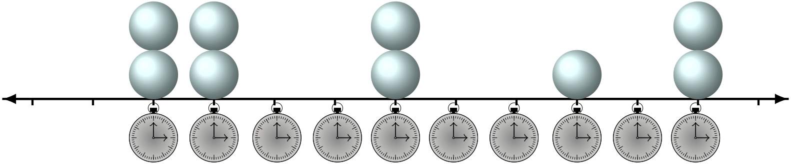

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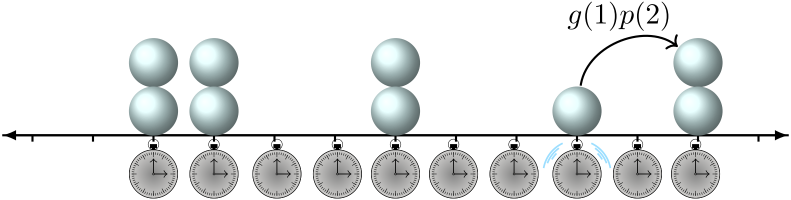

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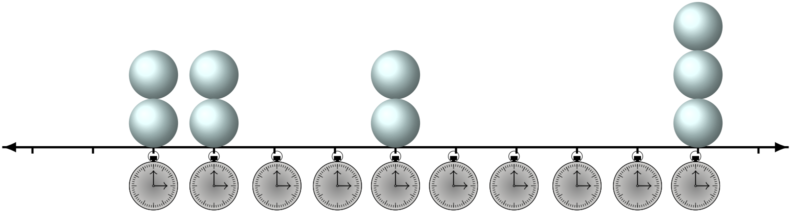
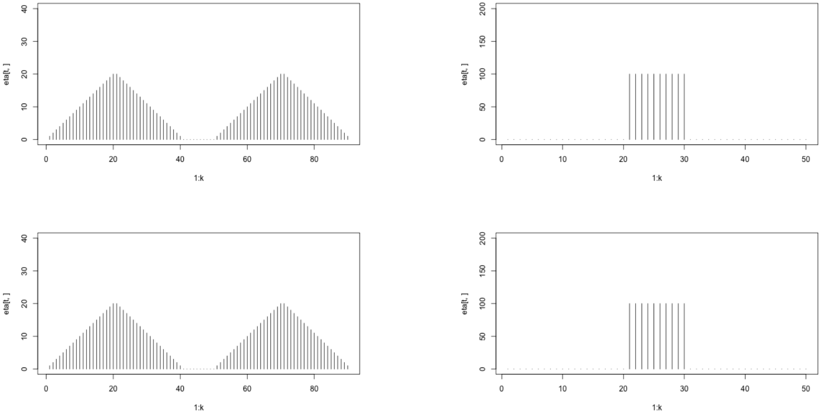


Simulations of Zero-Range: symmetric/asymmetric

Initial configurations:

$\eta_0 = (1, 2, 3, \ldots, 20, 19, 18, \ldots, 1, 0, \ldots, 0, 1, 2, 3, \ldots, 20, 19, 18, \ldots, 1)$ and $\xi_0 = (0, \cdots, 0, 100, \ldots, 100, 0, \ldots, 0)$

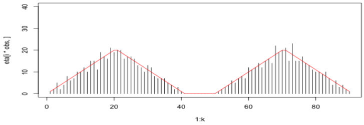
Upper displays: symmetric rates $p(1) = 0.5 = p(-1)$.

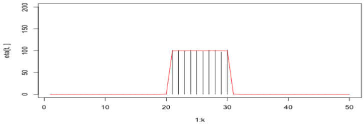


Lower displays: asymmetric rates $p(1) = 0.9,\ p(-1) = 0.1$.

Hydrodynamic Limit: empirical measure

Goal:
$\pi_t^N(\eta, du) \to_{N\to+\infty} \rho_t(u)du,$
where
$\pi_t^N(\eta, du) = \frac{1}{N}\sum_x \eta_t^N(x)\delta_{\frac{x}{N}}(du)$
and $\rho_t(\cdot)$ is solution of the hydrodynamic equation.





Possible hydrodynamic equations

Heat: $\partial_t\rho_t = \Delta\rho_t$ (p symmetric, tN^2)
Porous media: $\partial_t\rho_t = \Delta\rho_t^m, 2 \leq m \in \mathbb{N}$ (p symmetric, tN^2)
Inviscid Burgers: $\partial_t\rho_t = \nabla(\rho_t(1-\rho_t))$ (p asymmetric, tN)
Fractional heat: $\partial_t\rho_t = (-\Delta)^{\gamma/2}\rho_t$ ($p(\cdot) = \frac{c}{|\cdot|^{1+\gamma}}, tN^\gamma, \gamma\in(1,2)$)

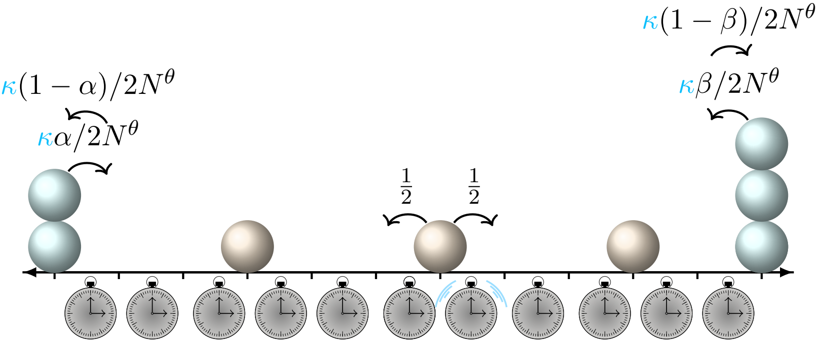
The focus of this presentation:

- I will present the hydrodynamic limit for an exclusion process in contact with stochastic reservoirs when jumps are long range given by a symmetric probability transition rate:
  - with finite variance;
  - with infinite variance.

Let us start with the simplest case: jumps to nearest-neighbors.

Now $\Lambda = [0,1]$ and $\Lambda_N = \{1, ..., N-1\}$.
The state space of the Markov process η_t^N is $\Omega_N = \{0,1\}^{\Lambda_N}$.

SSEP in contact with reservoirs:



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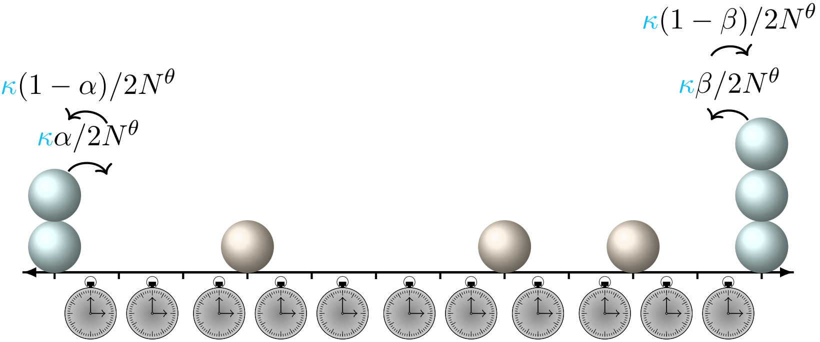

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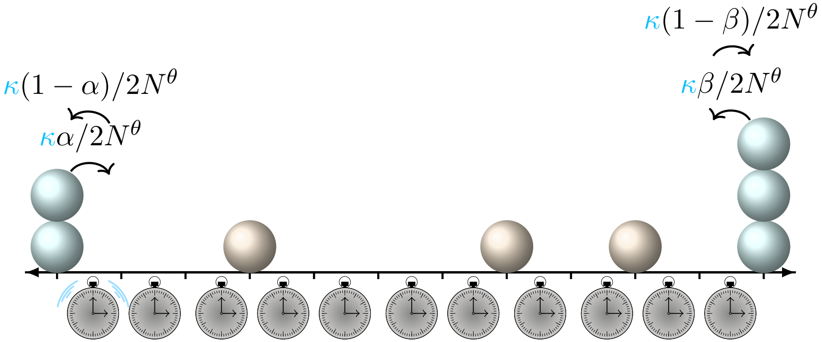

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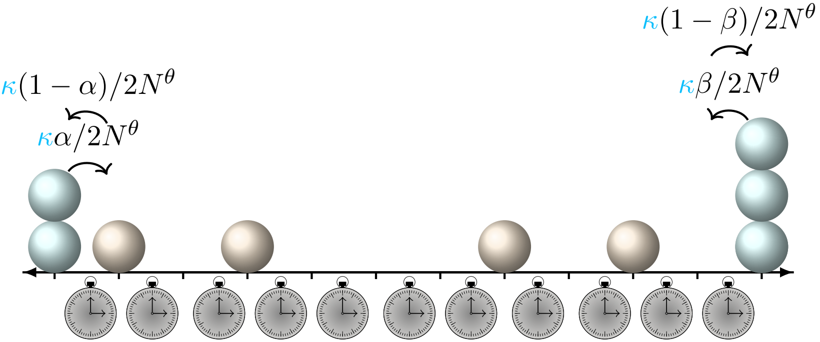

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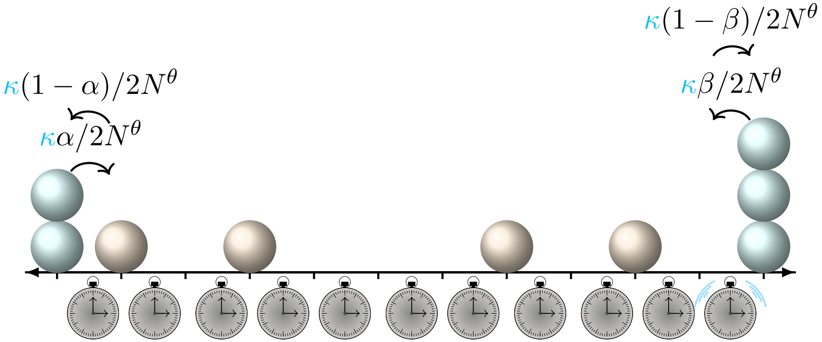

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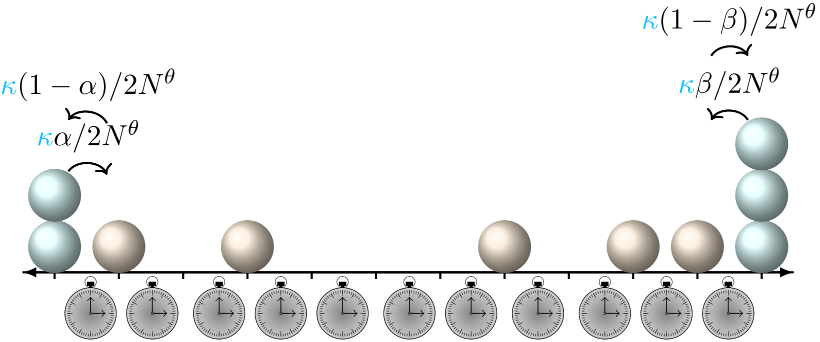


Invariant measures:

- **If $\alpha = \beta = \rho$ the Bernoulli product measures are invariant (equilibrium measures):**
  $\nu_\rho(\eta : \eta(x) = 1) = \rho.$

- **If $\alpha \neq \beta$ the Bernoulli product measure is no longer invariant, but since we have a finite state irreducible Markov process there exists a UNIQUE invariant measure: the stationary measure (non-equilibrium) denoted by μ_{ss}.**

- **By the matrix ansatz method one can get information about this measure.**
  (Not in the long jumps case.)

Hydrodynamic Limit:

♣ For $\eta \in \Omega_N$, let $\pi_t^N(\eta, dq) = \frac{1}{N}\sum_{x=1}^{N-1} \eta_{tN^2}(x)\delta_{x/N}(dq)$, be the *empirical measure. (Diffusive time scaling!)*

♣ Assumption: fix $g : [0,1] \to [0,1]$ measurable and probability measures $\{\mu_N\}_{N\geq 1}$ such that for every $H \in C([0,1])$,

$$\frac{1}{N}\sum_{x=1}^{N-1} H\left(\tfrac{x}{N}\right)\eta(x) \to_{N\to+\infty} \int_0^1 H(q)\, g(q)dq,$$

wrt μ_N. (μ_N is associated to $g(\cdot)$)

♣ Then: for any $t > 0$,

$$\pi_t^N(\eta, dq) \to_{N\to+\infty} \rho(t,q)dq,$$

wrt $\mu_N(t)$, where $\rho(t,q)$ evolves according to a PDE, the hydrodynamic equation.

Hydrodynamic Limit:

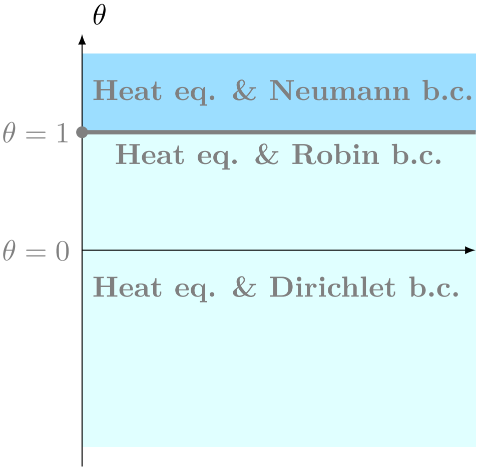
Theorem [Baldasso et al]:

Let $g:[0,1]\to[0,1]$ be a measurable function and let $\{\mu_N\}_{N\geq 1}$ be a sequence of probability measures in Ω_N associated to $g(\cdot)$. Then, for any $0\leq t\leq T$,

$$\lim_{N\to\infty}\mathbb{P}_{\mu_N}\Big(\Big|\frac{1}{N}\sum_{x=1}^{N-1}H(\tfrac{x}{N})\eta_{tN^2}(x)-\int_0^1 H(q)\rho(t,q)dq\Big|>\delta\Big)=0$$

and $\rho_t(\cdot)$ is the UNIQUE weak solution of the heat equation with different boundary conditions depending on the range of the parameter θ and with initial condition $g(\cdot)$.

Hydrodynamic equations:



Heat equation:
$\partial_t \rho_t(q) = \frac{1}{2}\partial_q^2 \rho_t(q).$

- $\theta > 1$ **Neumann b.c.:**
  $\partial_q \rho_t(0) = \partial_q \rho_t(1) = 0.$
- $\theta = 1$ **Robin b.c.:**
  $\partial_q \rho_t(0) = \kappa(\rho_t(0) - \alpha),$
  $\partial_q \rho_t(1) = \kappa(\beta - \rho_t(1)).$
- $\theta < 1$ **Dirichlet b.c.:**
  $\rho_t(0) = \alpha,\ \rho_t(1) = \beta.$

Hydrostatic Limit:

Theorem: Let μ_{ss} be the stationary measure for the process $\{\eta_{tN^2}\}_{t\geq 0}$. Then, μ_{ss} is associated to $\bar{\rho}:[0,1]\to[0,1]$ given on $q\in(0,1)$ by

$$\bar{\rho}(q)=\begin{cases}(\beta-\alpha)q+\alpha\,;\,\theta<1,\\ \frac{\kappa(\beta-\alpha)}{2+\kappa}q+\alpha+\frac{\beta-\alpha}{2+\kappa}\,;\,\theta=1,\\ \frac{\beta+\alpha}{2}\,;\,\theta>1,\end{cases}$$

$\bar{\rho}(\cdot)$ is a stationary solution of the hydrodynamic equation.

The proof:

How do we prove the results?

Two things to do:

- Tightness of $\mathbb{Q}_N$, where $\mathbb{Q}_N$ is induced by $\mathbb{P}_{\mu_N}$ and the map

$$\pi^N_\cdot : \mathcal{D}([0,T],\Omega_N) \longrightarrow \mathcal{D}([0,T],\mathcal{M}_+)$$

- Characterization of limit points: limit points are concentrated on trajectories of measures that are absolutely continuous wrt the Lebesgue measure and the density is a weak solution of the corresponding PDE:

$$\mathbb{Q}(\pi_\cdot : \pi_t(dq) = \rho(t,q)dq \text{ and } \rho_t(q) \text{ is solution to the PDE}) = 1.$$

Let us focus on last item.

The notion of weak solution:

Let $g:[0,1]\to[0,1]$ be measurable. We say $\rho:[0,T]\times[0,1]\to[0,1]$ is a weak solution to the heat equation with Dirichlet b.c. if:

- $\rho \in L^2(0,T;\mathcal{H}^1)$;
- ρ satisfies the weak formulation:

$$\int_0^1 \rho_t(q)H_t(q)-g(q)H_0(q)\,dq-\int_0^t\int_0^1 \rho_s(q)\Big(\tfrac{1}{2}\partial_q^2+\partial_s\Big)H_s(q)\,ds\,dq$$
$$+\frac{1}{2}\int_0^t \beta\partial_q H_s(1)-\alpha\partial_q H_s(0)\,ds=0,$$

for all $t\in[0,T]$ and any function $H\in C_0^{1,2}([0,T]\times(0,1))$.

Definition

The Sobolev space $\mathcal{H}^1$ on $(0,1)$ is the Hilbert space defined as the completion of $C^\infty([0,1])$ for the norm $\|\cdot\|^2_{\mathcal{H}^1}:=\|\cdot\|_2^2+\|\cdot\|_1^2$, where $\|H\|_1^2=\int_0^1(\partial_q H(q))^2\,dq$, The space $L^2(0,T;\mathcal{H}^1)$ is the set of measurable functions $f:[0,T]\to\mathcal{H}^1$ such that $\int_0^T\|f_s\|^2_{\mathcal{H}^1}\,ds<\infty$.

Other notion of solution:

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  for all $t\in[0,T]$ and any function $H\in C_c^{1,2}([0,T]\times(0,1))$;
- $\rho_t(0)=\alpha$ and $\rho_t(1)=\beta$, for $t\in(0,T]$.

How do we formulate the solution:

A simple computation shows that

$$
\begin{aligned}
N^2\mathscr{L}_N\langle\pi_s^N,H\rangle = \langle\pi_s^N,\tfrac{1}{2}\Delta_N H\rangle \\
&+\tfrac{1}{2}\nabla_N^+H(0)\eta_{sN^2}(1)-\tfrac{1}{2}\nabla_N^-H(1)\eta_{sN^2}(N-1)\\
&+\tfrac{\kappa}{2}N^{1-\theta}H\Big(\tfrac{1}{N}\Big)(\alpha-\eta_{sN^2}(1))\\
&+\tfrac{\kappa}{2}N^{1-\theta}H\Big(\tfrac{N-1}{N}\Big)(\beta-\eta_{sN^2}(N-1))
\end{aligned}
$$

If $H(0)=H(1)=0$, then from Dynkin's formula, we get

$$
\begin{aligned}
M_t^N(H) = \langle\pi_t^N,H\rangle-\langle\pi_0^N,H\rangle-\int_0^t\langle\pi_s^N,\tfrac{1}{2}\Delta_N H\rangle ds\\
&-\tfrac{1}{2}\int_0^t\nabla_N^+H(0)\eta_{sN^2}(1)-\nabla_N^-H(1)\eta_{sN^2}(N-1)ds+O(N^{-\theta}).
\end{aligned}
$$

How do we formulate the solution $\theta \in (0,1)$:

Replacing $\eta_{sN^2}(1)$ by α and $\eta_{sN^2}(N-1)$ by β ($\theta < 1!$) then

$$M_t^N(H) = \langle \pi_t^N, H\rangle - \langle \pi_0^N, H\rangle - \int_0^t \langle \pi_s^N, \tfrac{1}{2}\Delta_N H\rangle ds$$
$$- \tfrac{1}{2}\int_0^t \nabla_N^+ H(0)\alpha - \nabla_N^- H(1)\beta ds + O(N^{-\theta}).$$

Take the expectation and assuming that
$\rho_t^N(x) = \mathbb{E}_{\mu_N}[\eta_{tN^2}(x)] \sim \rho_t(x/N)$, for N big, we get

$$\int_0^1 \rho_t(q)H(q) - \rho_0(q)H(q)dq - \int_0^t\int_0^1 \tfrac{1}{2}\partial_q^2 H(q)\rho_s(q)dqds$$
$$- \tfrac{1}{2}\int_0^t \partial_q H(0)\alpha - \partial_q H(1)\beta ds = 0.$$

How do we formulate the solution $\theta \leq 0$:

Replacing $\eta_{sN^2}(1)$ by α and $\eta_{sN^2}(N-1)$ by β ($\theta < 1!$) then

$$M_t^N(H) = \langle \pi_t^N, H\rangle - \langle \pi_0^N, H\rangle - \int_0^t \langle \pi_s^N, \tfrac{1}{2}\Delta_N H\rangle ds$$
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$$- \tfrac{1}{2}\int_0^t \partial_q H(0)\alpha - \partial_q H(1)\beta ds = 0.$$

The discrete profile:

Fix an initial measure μ_N in Ω_N. For $x \in \Lambda_N$ and $t \geq 0$, let

$$\rho_t^N(x) \;=\; \mathbb{E}_{\mu_N}[\eta_{tN^2}(x)]\,.$$

We extend this definition to the boundary by setting

$$\rho_t^N(0) \;=\; \alpha \text{ and } \rho_t^N(N) \;=\; \beta\,, \text{ for all } t \geq 0\,.$$

A simple computation shows that $\rho_t^N(\cdot)$ is a solution of

$$\partial_t \rho_t^N(x) \;=\; N^2(\mathcal{B}_N \rho_t^N)(x)\,, \;\; x \in \Lambda_N\,, \;\; t \geq 0$$

where the operator $\mathcal{B}_N$ acts on functions $f : \Lambda_N \cup \{0, N\} \to \mathbb{R}$ as

$$\begin{aligned}
&N^2(\mathcal{B}_N f)(x) \;=\; \tfrac{1}{2}\Delta_N f(x), \quad \text{for } x \in \{2, \cdots, N-2\},\\
&N^2(\mathcal{B}_N f)(1) = N^2(f(2) - f(1)) + \tfrac{\kappa N^2}{N^\theta}(f(0) - f(1)),\\
&N^2(\mathcal{B}_N f)(N-1) = N^2(f(N-2) - f(N-1)) + \tfrac{\kappa N^2}{N^\theta}(f(N) - f(N-1)).
\end{aligned}$$

Stationary empirical profile:

The stationary solution of the previous equation is given by

$$\rho^N_{ss}(x) = \mathbb{E}_{\mu_{ss}}[\eta_{tN^2}(x)] = a_N x + b_N$$

where $a_N = \frac{\kappa(\beta-\alpha)}{2N^\theta+\kappa(N-2)}$ and $b_N = a_N(\frac{N^\theta}{\kappa} - 1) + \alpha$, so that

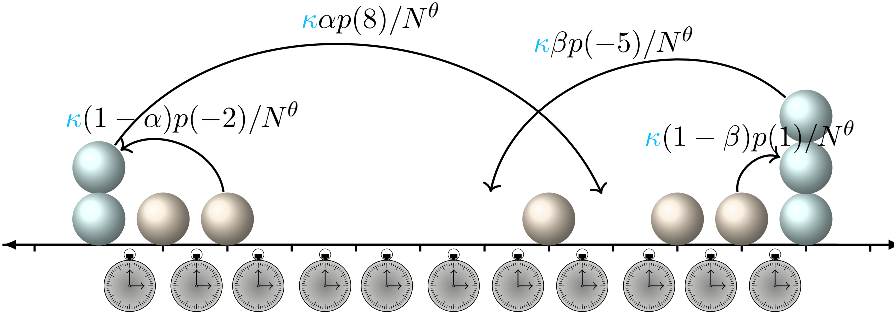
$$\lim_{N\to\infty} \max_{x\in\Lambda_N} \left|\rho^N_{ss}(x) - \bar{\rho}(\tfrac{x}{N})\right| = 0$$

where

$$\bar{\rho}(q) = \begin{cases} (\beta-\alpha)q+\alpha\,;\ \theta<1, \\ \frac{\kappa(\beta-\alpha)}{2+\kappa}q+\alpha+\frac{\beta-\alpha}{2+\kappa}\,;\ \theta=1, \\ \frac{\beta+\alpha}{2}\,;\ \theta>1, \end{cases}$$

is a stationary solution of the hydrodynamic equation.

Exclusion in contact with infinitely many reservoirs



What if jumps are arbitrarily big?

Let $p(\cdot)$ be a translation invariant transition probability given at $z \in \mathbb{Z}$ by

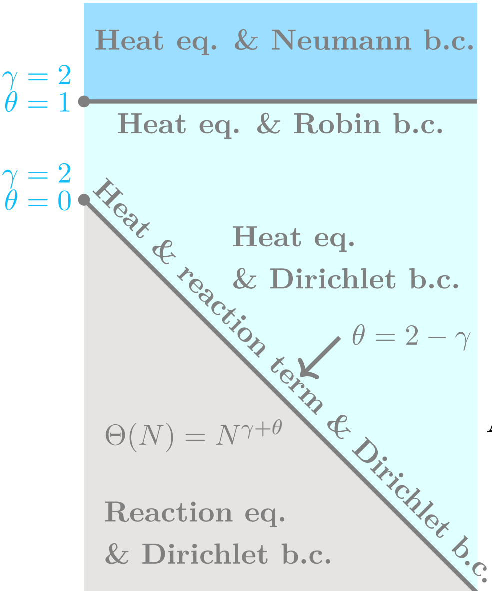
$$p(z) = \begin{cases} \frac{c_\gamma}{|z|^{\gamma+1}}, & z \neq 0, \\ 0, & z = 0, \end{cases}$$

where c_γ is a normalizing constant. Since $p(\cdot)$ is symmetric it is mean zero, that is:

$$\sum_{z \in \mathbb{Z}} z p(z) = 0$$

and take (by now) $\gamma > 2$ so that we define its variance by

$$\sigma_\gamma^2 = \sum_{z \in \mathbb{Z}} z^2 p(z) < \infty.$$



- **Heat equation:**

  $$\partial_t \rho_t(q) = \frac{\sigma^2}{2} \partial_q^2 \rho_t(q)$$

  - $\theta = 1$ **Robin b.c.:**

    $$\partial_q \rho_t(0) = \frac{2m\kappa}{\sigma^2}(\rho_t(0) - \alpha),$$
    $$\partial_q \rho_t(1) = \frac{2m\kappa}{\sigma^2}(\beta - \rho_t(1)),$$

- **Reaction-diffusion eq.:**

  $$\partial_t \rho_t(q) = \frac{\sigma^2}{2} \partial_q^2 \, \rho_t(q) + \kappa(V_0(q) - V_1(q)\rho_t(q))$$

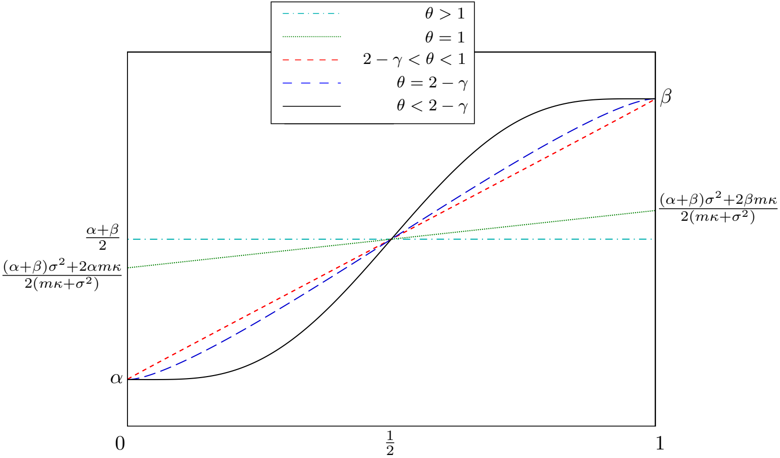
- **Reaction equation:**

  $$\partial_t \rho_t(q) = \kappa(V_0(q) - V_1(q)\rho_t(q))$$

Above

$$V_1(q) = \frac{c_\gamma}{\gamma}\Big(\frac{1}{q^\gamma} + \frac{1}{(1-q)^\gamma}\Big)$$
$$V_0(q) = \frac{c_\gamma}{\gamma}\Big(\frac{\alpha}{q^\gamma} + \frac{\beta}{(1-q)^\gamma}\Big).$$

Stationary solutions:



What about $\gamma \in (1,2)$?

We will get a collection of fractional reaction-diffusion equations

$$\partial_t \rho_t(q) = \mathbb{L}_{\kappa} \rho_t(q) + \kappa V_0(q).$$

where the operator $\mathbb{L}_{\kappa} = \mathbb{L} - \kappa V_1$, $\mathbb{L}$ is the regional fractional laplacian and

$$V_1(q) = \frac{c_\gamma}{\gamma}\Big(\frac{1}{q^\gamma} + \frac{1}{(1-q)^\gamma}\Big)$$

$$V_0(q) = \frac{c_\gamma}{\gamma}\Big(\frac{\alpha}{q^\gamma} + \frac{\beta}{(1-q)^\gamma}\Big).$$

The operator $\mathbb{L}$:

Let $(-\Delta)^{\gamma/2}$ be the fractional Laplacian of exponent $\gamma/2$ which is defined on the set of functions $H : \mathbb{R} \to \mathbb{R}$ such that

$$\int_{-\infty}^{\infty} \frac{|H(q)|}{(1+|q|)^{1+\gamma}} dq < \infty$$

by (provided the limit exists)

$$(-\Delta)^{\gamma/2} H\,(q) = c_\gamma \lim_{\varepsilon \to 0} \int_{-\infty}^{\infty} \mathbf{1}_{|u-q| \geq \varepsilon} \frac{H(q) - H(u)}{|u-q|^{1+\gamma}} du.$$

Let $\mathbb{L}$ be the regional fractional Laplacian on $[0,1]$, whose action on functions $H \in C_c^\infty(0,1)$ is given by

$$\begin{aligned}(\mathbb{L}H)(q) &= -(-\Delta)^{\gamma/2} H\,(q) + V_1(q) H(q) \\ &= c_\gamma \lim_{\varepsilon \to 0} \int_0^1 \mathbf{1}_{|u-q| \geq \varepsilon} \frac{H(u) - H(q)}{|u-q|^{1+\gamma}} du, \quad q \in (0,1).\end{aligned}$$

The fractional Sobolev space:

Definition

The Sobolev space $\mathcal{H}^{\gamma/2}$ consists of all square integrable functions $g:(0,1)\to\mathbb{R}$ such that $\|g\|_{\gamma/2}<\infty$, with

$$\|g\|_{\gamma/2} := \langle g,g\rangle_{\gamma/2} = \frac{c_\gamma}{2}\iint_{[0,1]^2}\frac{(g(u)-g(q))^2}{|u-q|^{1+\gamma}}dudq.$$

The space $L^2(0,T;\mathcal{H}^{\gamma/2})$ is the set of measurable functions $f:[0,T]\to\mathcal{H}^{\gamma/2}$ such that $\int_0^T \|f_t\|^2_{\mathcal{H}^{\gamma/2}}dt<\infty$ where $\|f_t\|^2_{\mathcal{H}^{\gamma/2}} := \|f_t\|^2 + \|f_t\|^2_{\gamma/2}$.

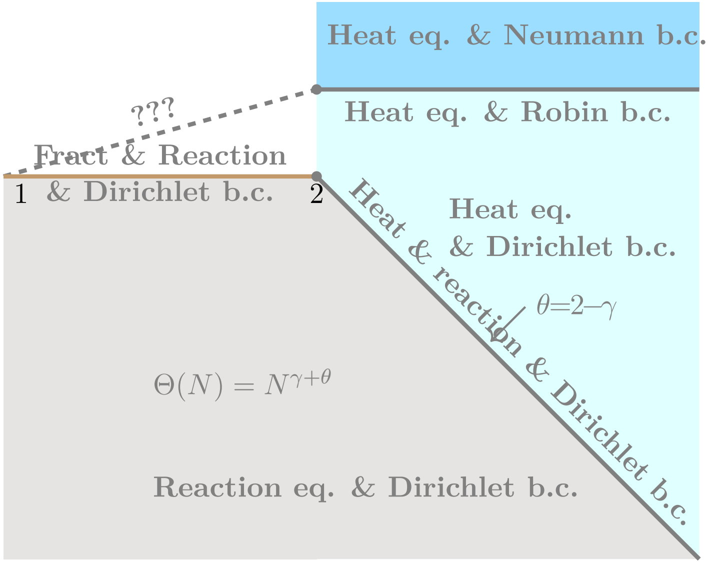
Weak solution of $\partial_t \rho_t(q) = \mathbb{L}_\kappa \rho_t(q) + \kappa V_0(q)$ with Dir.:

Let $g : [0,1] \to [0,1]$ be measurable. We say $\rho : [0,T] \times [0,1] \to [0,1]$ is a weak solution of the PDE above if:

- $\rho \in L^2(0,T;\mathcal{H}^{\gamma/2})$ and $\int_0^T \int_0^1 \left\{ \frac{(\alpha-\rho_t(q))^2}{q^\gamma} + \frac{(\beta-\rho_t(q))^2}{(1-q)^\gamma} \right\} dq\, dt < \infty$,
- For all $t \in [0,T]$ and any function $H \in C_c^{1,\infty}([0,T] \times (0,1))$:

$$\int_0^1 \rho_t(q) H_t(q) - g(q) H_0(q)\, dq - \int_0^t \int_0^1 \rho_s(q) \Big(\partial_s + \mathbb{L}_\kappa \Big) H_s(q)\, dq ds - \kappa \int_0^t \int_0^1 V_0(q) H_s(q) dq\, ds = 0.$$

Open problems:



Conjecture:

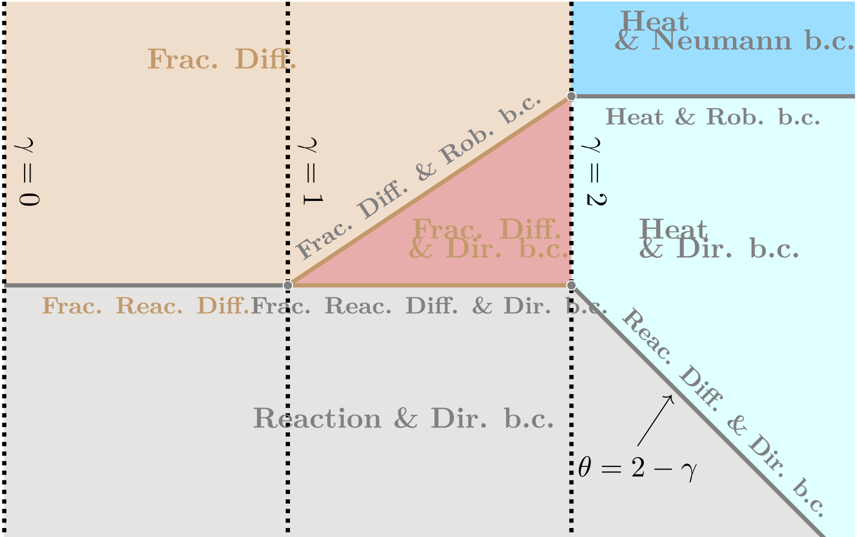
For $\theta > 0$ small and $\gamma \in (1,2)$ the solution should correspond to the solution when $\kappa = 0$. Supported by the result:

Let $g : [0,1] \to [0,1]$ be measurable and ρ^κ be the weak solution of

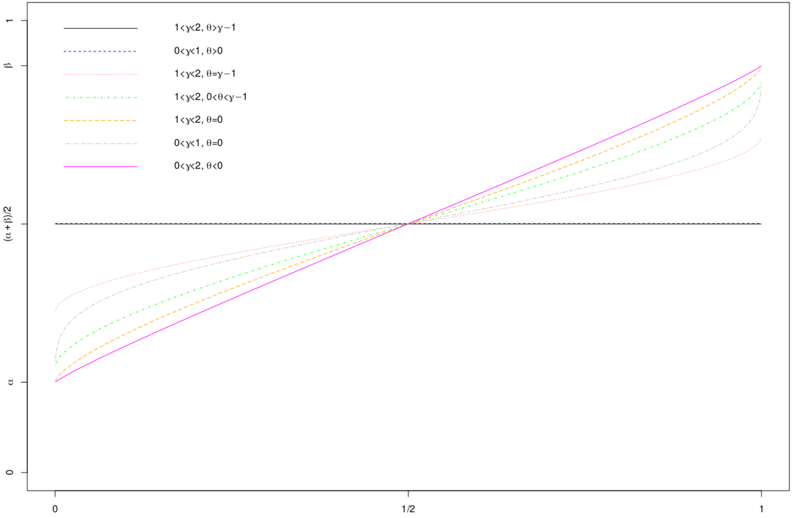
$$\partial_t \rho_t(q) = \mathbb{L}_\kappa \rho_t(q) + \kappa V_0(q),$$

with Dirichlet boundary conditions and initial condition $g(\cdot)$. Then ρ^κ converges strongly to ρ^0 in $L^2(0,T;\mathcal{H}^{\gamma/2})$ as κ goes to 0, where ρ^0 is the weak solution of the equation with $\kappa = 0$ and initial condition $g(\cdot)$.

Solved problem:



Stationary solutions:



For the future:

- What about hydrostatics?
- Fluctuations?
- Other boundary conditions?

Thank you very much!!!