①

A limiting case of the Hardy inequality and the perturbed Kolmogorov equation

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(III) Plan of my talk

§0. Topic of my research

§1. The Hardy ineq. ($p<N$)

- §1.0. Harmonic transplantation
- §1.1. Relation between the Hardy and the Sobolev ineq.
- §1.2. A limiting case of the Hardy ineq. ($p=N$)

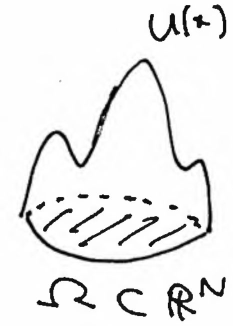
§2. Hardy's best constant and PDE

- §2.1. Known results
- §2.2. Kolmogorov equation

§0.

Let $1 \leq p < N$ and $p^* = \frac{Np}{N-p}$ $(>p)$

(Sobolev critical exponent)



Sobolev embedding ($p<N$)

$W_0^{1,p}(\Omega) \hookrightarrow L^{p^*}(\Omega)$: non-compact

Critical case ($p=N$)

$W_0^{1,N} \not\hookrightarrow L^\infty$

$W_0^{1,N} \hookrightarrow$?

$\Updownarrow$

Sobolev ineq. ($p<N$)

$${}^{\exists} C > 0 \quad C\|u\|_{p^*}^p \leq \|\nabla u\|_p^p \quad ({}^{\forall} u \in W_0^{1,p})$$

$\Updownarrow$

Minimization problem (associated with the best const.)

Explicit value? min. ?

$$S_{N,p} = \inf_{\substack{u \in W_0^{1,p} \\ u \not\equiv 0}} \frac{\int_\Omega |\nabla u|^p \, dx}{\left(\int_\Omega |u|^{p^*} dx \right)^{\frac{p}{p^*}}}$$

$\Updownarrow$

PDE (Euler-Lagrange eq.)

$$\begin{cases} -\Delta_p u = |u|^{p^*-2} u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2} \nabla u)$

② §1. $N=1$: Hardy, 1920, Hardy–Littlewood–Pólya, 1952

(I) The Hardy ineq. ($1 \le p < N$)

$$\boxed{\underbrace{\left(\frac{N-p}{p}\right)^p}_{\text{optimal}} \int_\Omega \frac{|u(x)|^p}{|x|^p}\,dx \ \underset{\text{if } u\not\equiv 0}{\underset{\neq}{\le}}\ \int_\Omega |\nabla u(x)|^p\,dx \quad \left(\forall u \in W_0^{1,p}(\Omega)\right)}$$

(∵) $$\int_\Omega \frac{|u|^p}{|x|^p} = \int_\Omega \frac{1}{N-p}\operatorname{div}\left(\frac{x}{|x|^p}\right)|u|^p = \frac{p}{N-p}\int_\Omega \frac{|u|^{p-2}u\,\nabla u\cdot x}{|x|^p}$$

Hölder's ineq. ⇒ $$\le \left(\frac{p}{N-p}\right)\left(\int_\Omega \frac{|u|^p}{|x|^p}\right)^{\frac{p-1}{p}}\left(\int \left|\nabla u\cdot\frac{x}{|x|}\right|^p\right)^{\frac{1}{p}}$$

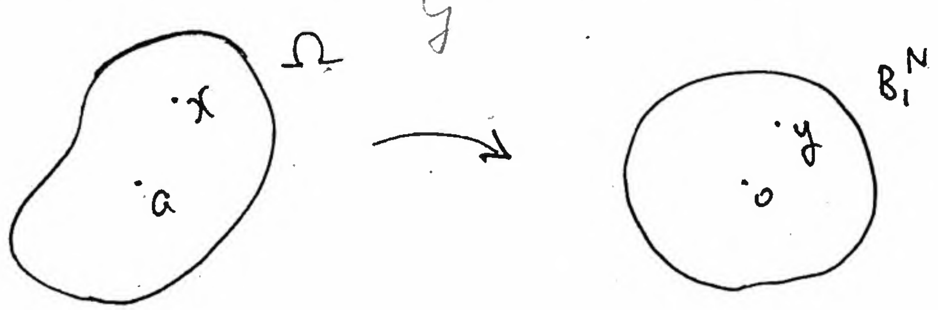
§1.0. $1 < p \le N$, $B_1^N = B_1 \subset \mathbb{R}^N$: the unit ball, $\Omega (\subset \mathbb{R}^N)$: domain,

$G_{\Omega,a}(\cdot)$: (p–) Green's funct. of Ω with singularity at $a \in \Omega$.
(or fundamental sol.)

(i.e. $\begin{cases} -\Delta_p G_{\Omega,a} = \delta_a & \text{in } \Omega, \\ G_{\Omega,a} = 0 & \text{on } \partial\Omega, \end{cases}$ δ_a : Dirac measure giving unit mass to the point a.)

For $v \in W^{1,p}_{0,rad}(B_1^N)$, define $u \in W_0^{1,p}(\Omega)$ as follows.

$$\boxed{u(x) = v\Big(\underbrace{(G_{B_1^N,0})^{-1}(G_{\Omega,a}(x))}_{y}\Big).}$$



Namely...

(II) Harmonic transplantation (Ref. Hersch, 1969)

$$\boxed{u(x) = v(y), \quad \text{where} \quad G_{\Omega,a}(x) = G_{B_1^N,0}(y)}$$

(Cf. The Riemann mapping theorem)

Note : $\|\nabla u\|_{L^p(\Omega)} = \|\nabla v\|_{L^p(B_1^N)}$

Applications : Moser, 1971, [Flucher, 1992, Csató–Roy, 2015] Bandle–Brillard–Flucher, 1998, Zographopoulos, 2010, Horiuchi–Kumlin, 2012, S.–Takahashi, 2017, S., 2019, Ioku, 2019 (§1.2 ($p \nearrow N$)), S., 2020 (§1.1 ($N \nearrow \infty$)), S., ArXiv, 2020 (higher order case), S., ArXiv, 2021 (Weighted case) ...

§1.1 Let $L^{p,q}$ be the Lorentz spaces. (※ $L^{p,p} = L^p$)

Fact :

The Hardy ineq. $\longleftrightarrow$ $W_0^{1,p} \hookrightarrow L^{p^*,\underline{p}}$

⇓ $\qquad \subsetneq$ (∵ $p^* > p$)

The Sobolev ineq. $\longleftrightarrow$ $W_0^{1,p} \hookrightarrow L^{p^*,\underline{p^*}} = L^{p^*}$

③ Let $p<N<m$. Consider (Ref. S., 2020)

$$\boxed{\underset{W^{1,p}_{0,rad}(\mathbb{R}^N)}{\underset{\cap}{u(x)}} = \underset{W^{1,p}_{0,rad}(\mathbb{R}^m)}{\underset{\cap}{v(y)}},\ \text{where}\ G_{\mathbb{R}^N,0}(x) = G_{\mathbb{R}^m,0}(y)}$$

i.e.

$$\left(\frac{p-1}{N-p}|S^{N-1}|^{-\frac{1}{p-1}}|x|^{-\frac{N-p}{p-1}} = \frac{p-1}{m-p}|S^{m-1}|^{-\frac{1}{p-1}}|y|^{-\frac{m-p}{p-1}}\right)$$

Then we have the followings.

The Sobolev ineq. on $\mathbb{R}^m$:

$$S_{m,p}\left(\int_{\mathbb{R}^m}|v|^{\frac{mp}{m-p}}dy\right)^{\frac{m-p}{m}} \leqq \int_{\mathbb{R}^m}|\nabla v|^p dy$$

$\downarrow (m\to\infty)$ $+\infty$ $\quad \|$ $\quad \|$

m^p

$$S_{m,p}\left(\frac{|S^{N-1}|}{|S^{m-1}|}\right)^{\frac{p}{m}}\left(\frac{N-p}{m-p}\right)^{p-\frac{p}{m}}\left(\int_{\mathbb{R}^N}\frac{|u|^{\frac{mp}{m-p}}}{|x|^{\frac{m-N}{m-p}p}}dx\right)^{\frac{m-p}{m}} \leqq \int_{\mathbb{R}^N}|\nabla u|^p dx$$

(Stirling's formula

$\Gamma(t) = \sqrt{2\pi}\, t^{t-\frac{1}{2}}e^{-t}+o(1)$

$(t\to\infty)$) $\Downarrow$ $(m\to\infty)$

The Hardy ineq. on $\mathbb{R}^N$:

$$\left(\frac{N-p}{p}\right)^p\int_{\mathbb{R}^N}\frac{|u|^p}{|x|^p}dx \leqq \int_{\mathbb{R}^N}|\nabla u|^p dx$$

Rem. The Sobolev ineq. $\Rightarrow$ The Hardy ineq.

(Cf. (Another way of "$m\to\infty$") Log-Sobolev ineq.

Beckner – Pearson, 1998. $\mathbb{R}^m \cong \underbrace{\mathbb{R}^N\times\cdots\times\mathbb{R}^N}_{\ell\ \text{times}}$

$m = N\ell$ $\left(\begin{smallmatrix} m\to\infty \\ \Leftrightarrow \ell\to\infty\end{smallmatrix}\right)$)

§1.2 Let $p<N$. Consider (Ref. Ioku, '19)

$$\boxed{\underset{W^{1,p}_{0,rad}(\mathbb{R}^N)}{\underset{\cap}{u(x)}} = \underset{W^{1,p}_{0,rad}(B_1^N)}{\underset{\cap}{v(y)}},\ \text{where}\ G_{\mathbb{R}^N,0}(x) = G_{B_1^N,0}(y)}$$

i.e.

$$\left(|x|^{-\frac{N-p}{p-1}} = |y|^{-\frac{N-p}{p-1}} - 1\right)$$

Then we have the followings. $\infty = \int\frac{|u|^N}{|x|^N}$ in general.

The Hardy ineq. on $\mathbb{R}^N$ $(p<N)$:

$$\underbrace{\left(\frac{N-p}{p}\right)^p}_{\downarrow (p\nearrow N)\ 0}\int_{\mathbb{R}^N}\frac{|u|^p}{|x|^p}dx \leqq \int_{\mathbb{R}^N}|\nabla u|^p dx$$

$\|$ $\quad \|$

$$\left(\frac{N-p}{p}\right)^p\int_{B_1^N}\frac{|v|^p}{|y|^p\underbrace{\left(1-|y|^{\frac{N-p}{p-1}}\right)^p}_{\left(\frac{N-p}{p-1}\right)^p\left(\log\frac{1}{r}\right)^p}}dy \leqq \int_{B_1^N}|\nabla v|^p dy$$

($1-r^a = a\log\frac{1}{r}+o(1)$ $(a\to 0)$) $\Downarrow$ $(p\nearrow N)$

Ⅲ The critical Hardy ineq. $(p=N)$

$$\boxed{\underbrace{\left(\frac{N-1}{N}\right)^N}_{\text{optimal}}\int_{B_1^N}\frac{|v|^N}{|y|^N\left(\log\frac{1}{|y|}\right)^N}dy \lneqq \int_{B_1^N}|\nabla u|^N dy \quad \text{if } u\not\equiv 0}$$

(Cf. (Another way of "$p\nearrow N$" + Poincaré ineq.)
"$|\Omega|\to 0$"

S. – Sobukawa, 2020.)

$\lambda_p^1(\Omega)\int_\Omega|u|^p = \int_\Omega|\nabla u|^p$

④ **§2.** The Hardy ineq. appears in

- Stability of singular sol. to $\begin{cases} -\Delta u = u^p & \text{(Lane–Emden)} \\ -\Delta u = e^u & \text{(Liouville)} \end{cases}$
- Existence and nonexistence of sol. to heat eq. perturbed by singular potential.

§2.1 (P) $\begin{cases} \partial_t u(x,t) = \Delta u(x,t) + \frac{c}{|x|^2} u(x,t), & t>0, x \in \mathbb{R}^N \\ u(x,0) = u_0(x), & x \in \mathbb{R}^N. \end{cases}$

Thm A (Baras – Goldstein, 1984)

Let $0 \leqq u_0 \in L^2(\mathbb{R}^N)$.

(i) If $c \leqq \left(\frac{N-2}{2}\right)^2$, then (P) has a positive sol.

(ii) If $c > \left(\frac{N-2}{2}\right)^2$, then (P) has no positive sol. at all positive times.

"instantaneous blow-up"

⑪ Various extensions:

- Azorero – Alonso, 1998 ($\Delta \to \Delta_p$)
- Cabre – Martel, 1999 ($\frac{c}{|x|^2} \to V(x)$)
- G.R. Goldstein – J.A. Goldstein – Rhandi, 2012 ($\Delta \to \Delta - Ax\cdot\nabla$) ($N \geqq 3$)
- Ishige – Ishiwata, 2012 $\left(\begin{cases} \frac{\partial u}{\partial t} = \Delta u & \text{in } \mathbb{R}^N_+ \times (0,T) \\ \frac{\partial u}{\partial x_N} = \frac{\omega}{|x|} u & \text{on } \partial \mathbb{R}^N_+ \times (0,T) \end{cases} \right)$
- Yanagida et. al. (yesterday's talk!)

etc.

Q. N=2?

§2.2. $\rho(x) > 0$, $\alpha \in (0,1)$, $\rho \in C^{1,\alpha}_{loc}(\mathbb{R}^N)$, $\int_{\mathbb{R}^N} \rho(x)\,dx = 1$.

$$Lu := \Delta u + \frac{\nabla \rho}{\rho} \cdot \nabla u$$: Kolmogorov operator w.r.t. $d\mu = \rho(x)\,dx$.

Rem 1. $$\int_{\mathbb{R}^N} \nabla u \cdot \nabla v \, d\mu = -\int_{\mathbb{R}^N} (Lu) v \, d\mu.$$

Rem 2: If $\rho(x) = \rho_A(x) = c \cdot \exp\left(-\frac{1}{2}(x^t A x)\right)$,

A: $N\times N$ – symmetric positive definite matrix.

then $\frac{\nabla \rho}{\rho} = -Ax$.

$L_A u = \Delta u - Ax \cdot \nabla u$: Symmetric Ornstein–Uhlenbeck operator

Thm B (G.–G.–R., 2012)

Let $N \geqq 3$, $d\mu_A = \rho_A(x)\,dx$, and $0 \leqq V(x) \leqq \frac{c}{|x|^2}$ $(x \in \mathbb{R}^N)$.

(i) If $c \leqq \left(\frac{N-2}{2}\right)^2$, then $0 \leqq \exists u \in C([0,\infty); L^2(\mathbb{R}^N; d\mu_A))$

: weak sol. of $(\tilde{P})$ $\begin{cases} \partial_t u = L_A u + V(x) u, & t>0, x \in \mathbb{R}^N \\ u(x,0) = u_0(x), & x \in \mathbb{R}^N. \end{cases}$

satisfying (EB) $\|u(t)\|_{L^2(\mathbb{R}^N, d\mu_A)} \leqq M e^{\omega t} \|u_0\|_{L^2(\mathbb{R}^N; d\mu_A)}$ $(t \geqq 0)$

for some constants $M \geqq 1$, $\omega \in \mathbb{R}$ and for any $u_0 \in L^2(\mathbb{R}^N; d\mu_A)$.

(ii) If $c > \left(\frac{N-2}{2}\right)^2$, then for any $0 \leqq u_0 \in L^2(\mathbb{R}^N; d\mu_A)$ ($\not\equiv 0$), there is no positive weak sol. ~~to~~ $(\tilde{P})$ with $V(x) = \frac{c}{|x|^2}$ satisfying (EB).

⑤ $0\in\Omega\subset\mathbb{R}^2$: b'dd domain, $R=\sup_{x\in\Omega}|x|$.

$$(K_V)\quad\begin{cases}\partial_t u = L\, u + V(x)\,u, & t>0,\ x\in\Omega\\ u=0, & t>0,\ x\in\partial\Omega\\ u(x,0)=u_0(x), & x\in\Omega\end{cases}$$

Def (Weak sol. to (K_V))

u : weak sol. to (K_V)

$\overset{\text{def}}{\Longleftrightarrow}$ For each $T>0$ and any compact subset $K\subset\Omega$,

$u\in C([0,T]:L^2(\Omega:d\mu))$, $Vu\in L^1(K\times(0,T),d\mu dt)$

and
$$\int_0^T\int_\Omega u(-\partial_t\phi-L\phi)\,d\mu dt-\int_\Omega u_0\phi(\cdot,0)\,d\mu=\int_0^T\int_\Omega Vu\phi\,d\mu dt$$
for any $\phi\in W_2^{2,1}(Q_T)$ s.t. $\phi(\cdot,t)$ has compact support in Ω and $\phi(\cdot,T)=0$, where $Q_T=\Omega\times(0,T)$ and
$$W_{\underset{L^2}{2}}^{\overset{D_x}{2},\overset{\partial_t}{1}}(Q_T)=\{u\in L^2(Q_T)\mid D_x^\alpha u\in L^2(Q_T)\text{ for }|\alpha|\le 2,\ \partial_t u\in L^2(Q_T)\}$$

Thm (S.-Takahashi, 2019)

Let $L=L_A$, $d\mu_A=\rho_A(x)\,dx$, and $0\le V(x)\le \dfrac{c}{|x|^2(\log\frac{R}{|x|})^2}$

(i) If $c\le\frac14$, then $0\le{}^\exists u\in C([0,\infty):L^2(\Omega:d\mu_A))$

: weak sol. of (K_V) satisfying

(EB) $\|u(t)\|_{L^2(\Omega;d\mu_A)}\le Me^{\omega t}\|u_0\|_{L^2(\Omega:d\mu_A)}$

(ii) If $c>\frac14$, then for any $0\le u_0\in L^2(\Omega;d\mu_A)\setminus\{0\}$, there is no positive weak sol. of (K_V) with $V(x)=\dfrac{c}{|x|^2(\log\frac{R}{|x|})^2}$ satisfying (EB).

Lemma 1

Let $\Omega\subset\mathbb{R}^2$: b'dd, $R=\sup_{x\in\Omega}|x|$, and $d\mu_A=\rho_A(x)\,dx$.

Then the ineq.
$$\frac14\int_\Omega\frac{|u|^2}{|x|^2(\log\frac{R}{|x|})^2}\,d\mu_A\le\int_\Omega\left|\nabla u\cdot\frac{x}{|x|}\right|^2d\mu_A+\overset{\tilde\lambda}{\overset{\shortparallel}{\frac12}}\int_\Omega\frac{|u|^2(x^tAx)}{|x|^2(\log\frac{R}{|x|})}\,d\mu_A$$
holds for any $u\in W_0^{1,2}(\Omega:d\mu_A)$.

Moreover, if $0\in\Omega$, then $\frac14$ is optimal.

Rem Let $\tilde\lambda\ge0$ and $\lambda>\frac14$. Then
$$\inf_{\substack{u\in W_0^{1,2}(\Omega;d\mu_A)\\ u\not\equiv0}}\frac{E_\lambda(u):=\int_\Omega|\nabla u|^2d\mu_A+\tilde\lambda\int_\Omega\frac{|u|^2(x^tAx)}{|x|^2(\log\frac{R}{|x|})}\,d\mu_A-\lambda\int_\Omega\frac{|u|^2}{|x|^2(\log\frac{R}{|x|})^2}\,d\mu_A}{\int_\Omega|u|^2d\mu_A}$$
$$=-\infty.$$

⑥ Define

$$\lambda_1(L+V) := \inf_{\substack{\phi\in W_0^{1,2}(\Omega;d\mu)\\ \phi\not\equiv 0}} \frac{\int_\Omega |\nabla\phi|^2 d\mu - \int_\Omega V\phi^2 d\mu}{\int_\Omega \phi^2 d\mu}$$

Lemma 2

Assume that $0<\rho\in C^1(\Omega)\cap C(\bar\Omega)$ and $0\le V\in L^1_{loc}(\Omega)$.

(i) If $\lambda_1(L+V)>-\infty$, then for any $0\le u_0\in L^2(\Omega;d\mu)$, there exists a positive weak sol. $u\in C([0,\infty),L^2(\Omega;d\mu))$ of (K_V) satisfying (EB).

(ii) If $\lambda_1(L+V)=-\infty$, then for any $0\le u_0\in L^2(\Omega;d\mu)\setminus\{0\}$ there is no positive weak sol. of (K_V) satisfying (EB).

Lemma 1 + Lemma 2 $\Longrightarrow$ Thm.

Proof of Lemma 1 For $x\in\Omega\setminus\{0\}$,

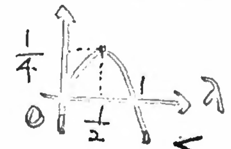
$$\operatorname{div}\Big[\underbrace{\lambda\rho_A(x)\frac{x}{|x|^2(\log\frac{R}{|x|})}}_{=:\ \mathbb{F}}\Big] = \lambda\rho_A(x)\Big[\underbrace{\frac{1}{|x|^2(\log\frac{R}{|x|})^2} - \frac{(x^tAx)}{|x|^2(\log\frac{R}{|x|})}}_{\in L^1_{loc}(\Omega)}\Big].$$

$$\int_\Omega |u|^2\lambda\Big[\frac{1}{|x|^2(\log\frac{R}{|x|})^2} - \frac{(x^tAx)}{|x|^2(\log\frac{R}{|x|})}\Big]d\mu_A$$

$$= -2\int_\Omega u(\nabla u\cdot\mathbb{F})\,dx = -2\lambda\int_\Omega \frac{u}{|x|(\log\frac{R}{|x|})}\Big(\nabla u\cdot\frac{x}{|x|}\Big)d\mu_A$$

$$\le \int_\Omega\Big|\nabla u\cdot\frac{x}{|x|}\Big|^2 d\mu_A + \lambda^2\int_\Omega \frac{|u|^2}{|x|^2(\log\frac{R}{|x|})^2}d\mu_A$$

$$\therefore\ (\lambda-\lambda^2)\int_\Omega\frac{|u|^2}{|x|^2(\log\frac{R}{|x|})^2}d\mu_A$$



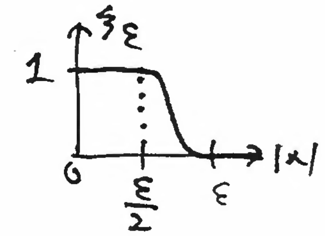
$$\le \int_\Omega\Big|\nabla u\cdot\frac{x}{|x|}\Big|^2 d\mu_A + \lambda\int_\Omega\frac{|u|^2(x^tAx)}{|x|^2(\log\frac{R}{|x|})}d\mu_A$$

If we choose $\lambda=\frac12$, then we obtain the ineq. ┘

(Optimality) Fix $\lambda>\frac14$ and $\tilde\lambda\ge 0$. Consider

$$\varphi_{\gamma,\varepsilon}(x) = \Big(\log\frac{R}{|x|}\Big)^{\gamma}\xi_\varepsilon(x),$$

where $\gamma<\frac12$, $\varepsilon>0$, $B(\varepsilon)\subset\Omega$, $\xi_\varepsilon\in C_c^\infty(B(\varepsilon))$.



Then $E_\lambda(\varphi_{\gamma,\varepsilon})$ $(\alpha>0)$

$$\le c\pi\Big(\log\frac{2R}{\varepsilon}\Big)^{2(\gamma-\frac12)}\underbrace{\Big(\frac12-\gamma\Big)^{-1}}_{\to\infty}\Big[\underbrace{\gamma^2-\lambda\exp\Big(-\frac{\alpha}{2}\Big(\frac{\varepsilon}{2}\Big)^2\Big)}_{<0\ \text{for small}\ \varepsilon}\Big] + o\Big(\Big(\frac12-\gamma\Big)^{-1}\Big)\quad \Big(\gamma\nearrow\frac12\Big)$$

$\to -\infty$

$\therefore\ \frac14$ is optimal. //

(7) Proof of Lemma 2 (Ref. B.-G., 1984, C.-M., 1999, G.-G.-R., 2012)

(i) $\lambda_1(L+V) > -\infty$, $u_0 \geq 0$, $u_0 \not\equiv 0$, $n \in \mathbb{N}$.

$V_n(x) := \min\{V(x), n\}$, $u_{0,n}(x) = \min\{u_0(x), n\}$

$$(K_{V_n}) \begin{cases} \partial_t u_n = L u_n + V_n(x) u_n, & t>0,\ x\in\Omega \\ u_n = 0, & t>0,\ x\in\partial\Omega \\ u_n(x,0) = u_{0,n}(x), & x\in\Omega \end{cases}$$

(Ref. Lorenzi – Bertoldi, 2007 (book))

$\Longrightarrow$ · (K_{V_n}) admits a unique positive classical sol. u_n

· $0 < u_n(x,t) \leq u_{n+1}(x,t)$ for $n\in\mathbb{N}$ and $(x,t)\in\Omega\times(0,\infty)$

$\int_\Omega (K_{V_n})\cdot u_n \, d\mu$

$$\Longrightarrow \frac{1}{2}\int_\Omega \partial_t(u_n^2)\,d\mu = -\int_\Omega |\nabla u_n|^2 d\mu + \int_\Omega \underbrace{V_n}_{\leq V} u_n^2\, d\mu$$

$$\leq -\lambda_1(L+V)\int_\Omega u_n^2(t)\,d\mu$$

$$\Longrightarrow \|u_n(t)\|_{L^2(\Omega:d\mu)} \leq e^{-\lambda_1(L+V)t}\|u_{0,n}\|_{L^2(\Omega:d\mu)}$$

$$\leq \underbrace{e^{-\lambda_1(L+V)t}\|u_0\|_{L^2(\Omega:d\mu)}}_{\text{independent of } n} \quad (t\geq 0)$$

monotone convergence thm.

$\Longrightarrow$ $u_n(t)\to u(t)$ in $L^2(\Omega;d\mu)$ uniformly for $t\in[0,T]$

u is a weak sol. of (K_V).

(ii) $\lambda_1(L+V) = -\infty$.

Assume $\exists u$: a positive sol. of (K_V) with initial data $0\leq u_0 \in L^2(\Omega:d\mu)\setminus\{0\}$ satisfying (EB)

Fix $\phi\in C_c^\infty(\Omega)$ with $\int_\Omega \phi^2\,d\mu = 1$.

$$(K_n) \begin{cases} \partial_t v_n = L v_n, & t>0,\ x\in\Omega \\ v_n = 0, & t>0,\ x\in\partial\Omega \\ v_n(x,0) = u_{0,n}(x), & x\in\Omega. \end{cases}$$

Then

$$\underbrace{u(x,t)}_{\text{sol. of } (K_V)} \geq u_n(x,t) \geq v_n(x,t) \geq v_1(x,t) \quad (t\geq 0)$$

$u_n(x,t) \downarrow \tilde{u}(x,t)$: minimal sol. of (K_V)

(※ $v_1(x,t)>0$ for a.e. $x\in\operatorname{supp}\phi$. $\Rightarrow$ $u_n(x,t)\geq v_1(x,t)>0$.)

$\int_\Omega (K_{V_n})\cdot \frac{\phi^2}{u_n}\,d\mu$

$$\Longrightarrow \int_\Omega V_n\phi^2\,d\mu = \int_\Omega \frac{\partial_t u_n}{u_n}\phi^2 + \nabla u_n\cdot\nabla\left(\frac{\phi^2}{u_n}\right)d\mu$$

$$\leq \partial_t\left(\int_\Omega (\log u_n)\phi^2\,d\mu\right) + \int_\Omega |\nabla\phi|^2 d\mu$$

$\int_1^t (\cdots)\,dt$ & $n\to\infty$

$$\Longrightarrow \int_\Omega V\phi^2\,d\mu - \int_\Omega|\nabla\phi|^2 d\mu \leq \frac{1}{t-1}\int_\Omega \log\frac{\tilde{u}(t)}{\tilde{u}(1)}\phi^2\,d\mu$$

Jensen's ineq. & (EB)

$$\Longrightarrow \int_\Omega V\phi^2\,d\mu - \int_\Omega|\nabla\phi|^2 d\mu \leq \frac{1}{2(t-1)}\Big\{2\log\left(M\|u_0\|_{L^2(\Omega:d\mu)}\right) + 2\omega t + 2\log\|\phi\|_\infty - 2\int_\Omega(\log\tilde{u}(1))\phi^2\,d\mu\Big\}$$

$\xrightarrow{t\to\infty}$ $\omega < \infty$ which contradicts $\lambda_1(L+V) = -\infty$.