Nonlinear fluctuating hydrodynamics for stochastic interacting particle systems

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0. General remarks from physics perspective

- PDE and probability: deterministic and fluctuating
- Thermodynamics (macro) and statistical mechanics (micro) for equilibrium systems.

  Fluctuations (probabilistic) only in the latter.
- For non-equilibrium systems, space-time dependent macroscopic description is hydrodynamics (PDE).

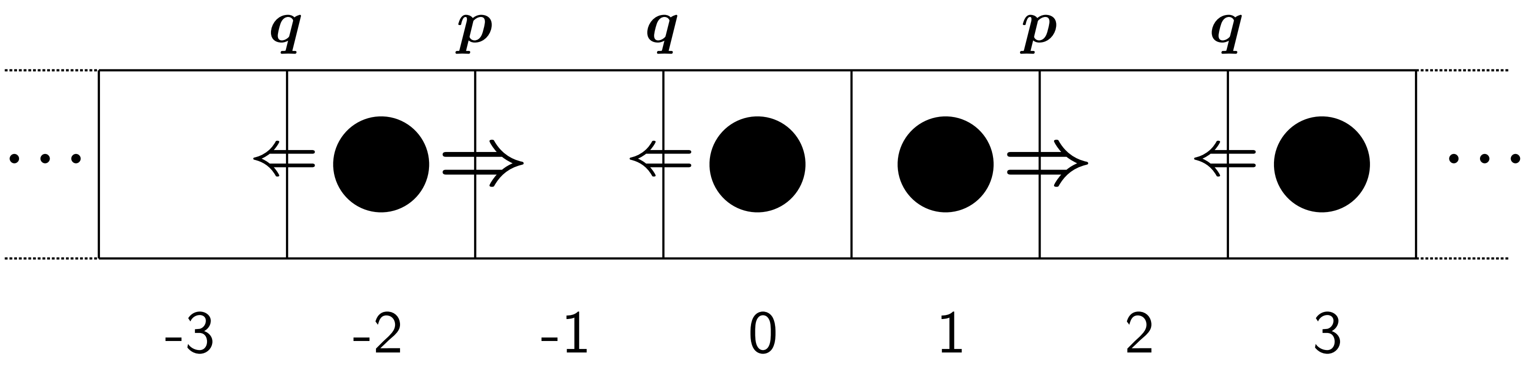
  Also interested in non-equilibrium fluctuations.
- Underlying dynamics can be taken to be Hamiltonian (deterministic) or Markov (stochastic).

Plan

1. Basics for single conserved quantity: ASEP and KPZ
2. Nonlinear fluctuating hydrodynamics
3. NLFHD for stochastic particle systems
4. Exact confirmation for two species exclusion process

1. Introduction: ASEP and KPZ universality

ASEP = asymmetric simple exclusion process

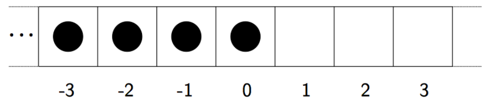


- Discrete Markov process. Non-equilibrium microscopic model.
- Bernoulli (independent sites) with density ρ is stationary.
- Hydrodynamics: Burgers eq. for density field $u = u(x, t)$
$$\frac{\partial}{\partial t} u + (p - q) \frac{\partial}{\partial x} u(1 - u) = 0.$$
Note the average current $j(\rho) = (p - q)\rho(1 - \rho)$.
- $N(x, t)$: Integrated current at $(x, x + 1)$ upto time t

Current fluctuations

2000 Johansson

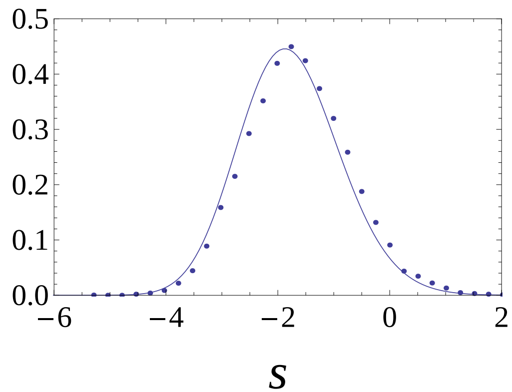
Th. For TASEP ($q=0$) with the step i.c.



$$\lim_{t\to\infty}\mathbb{P}\left[\frac{N(0,t)-t/4}{-2^{-4/3}t^{1/3}}\le s\right]=F_2(s)$$

where $F_2(s)$ is the GUE Tracy-Widom distribution

$$F_2(s)=\det(1-P_sK_{\mathrm{Ai}}P_s)_{L^2(\mathbb{R})}$$



where P_s: projection onto the interval $[s,\infty)$
and K_{Ai} is the Airy kernel

$$K_{\mathrm{Ai}}(x,y)=\int_0^\infty d\lambda \mathrm{Ai}(x+\lambda)\mathrm{Ai}(y+\lambda)$$

Rem: Generalization to ASEP is available (Tracy-Widom 2009).

Gaussian Unitary Ensemble (GUE)

Measure for N-dim hermitian matrices H

$$\frac{1}{Z}\exp[-\mathrm{Tr}\ H^2]dH$$

Eigenvalues (x_i) density for GUE (Product of two determinants)

$$\frac{1}{Z}\prod_{i<j}(x_i - x_j)^2 \prod_i e^{-x_i^2}$$

Largest eigenvalue distribution (GUE TW for $N \to \infty$)

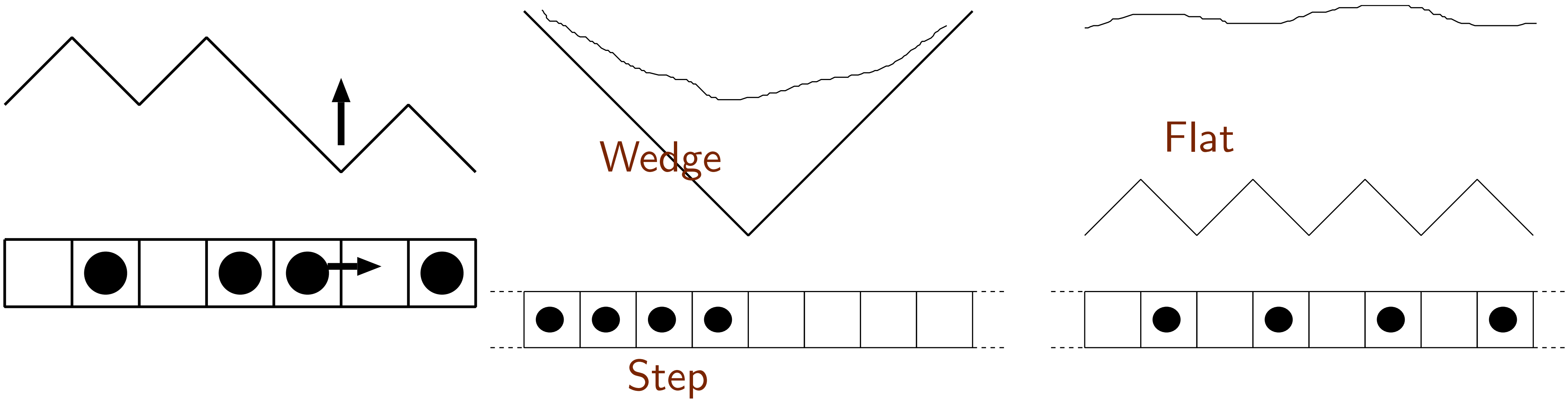
$$\mathbb{P}[x_{\mathrm{max}} \le s] = \det(1 - P_s K_N P_s)_{L^2(\mathbb{R})} \quad (\Rightarrow F_2)$$

$$K_N(x,y) = \sum_{n=0}^{N-1} c_n H_n(x) H_n(y) e^{-(x^2+y^2)/2}$$

GOE TW for real symmetric matrix ensemble.

Surface growth related to ASEP

Mapping to a surface growth model (single step model)



Current fluctuation $\Leftrightarrow$ Height fluctuation

Shows universal behaviors.

KPZ equation

$h(x,t)$: height at position $x \in \mathbb{R}$ and at time $t \geq 0$

1986 Kardar Parisi Zhang (SPDE)

$$\partial_t h(x,t) = \frac{1}{2}(\partial_x h(x,t))^2 + \frac{1}{2}\partial_x^2 h(x,t) + \eta(x,t)$$

where η is the Gaussian noise with mean 0 and covariance

$$\langle \eta(x,t)\eta(x',t')\rangle = \delta(x-x')\delta(t-t')$$

- Can be obtained by a weak asymmetry limit of ASEP.
- Well-definedness of the equation ($\Rightarrow$ Hairer, etc)
- Th. (TS-Spohn, Amir et al 2010)
  Height at $x = 0$: $h(x=0,t) = vt + ct^{1/3}\chi_t$
  χ_t tends to TW when $t \to \infty$ (KPZ universality class)

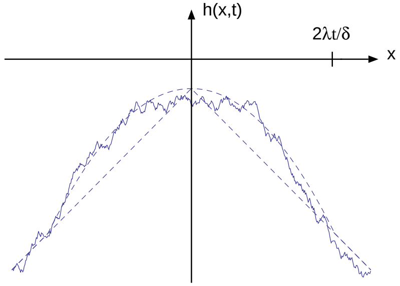
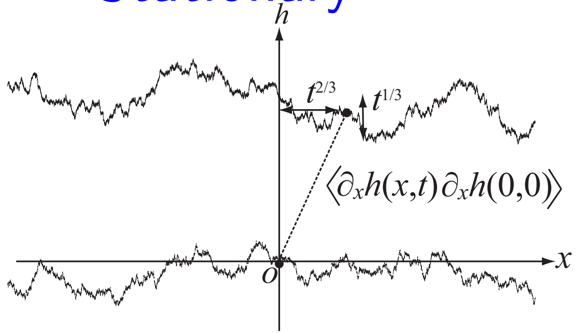
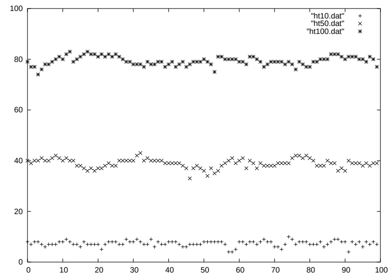
Noisy Burgers equation

- For $u(x,t) = \partial_x h(x,t)$, the KPZ equation becomes the noisy Burgers equation
$$\partial_t u = \frac{1}{2}\partial_x^2 u + \frac{1}{2}\partial_x u^2 + \partial_x \eta$$
This is obtained by adding noise $\partial_x \eta$ to the Burgers equation.

- Remark: Weakly asymmetric limit can not be taken for TASEP. But the correct universality can be captured by adding noise to its hydrodynamic equation (Burgers).

- This may be considered as the simplest example of fluctuating hydrodynamics for the case of single conserved quantity.

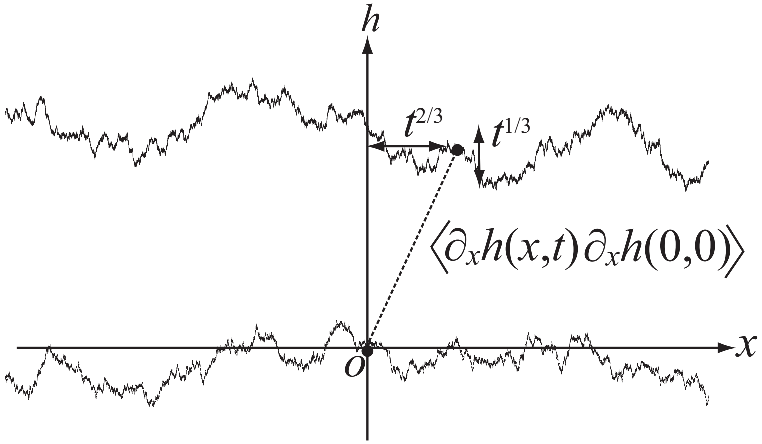
3 important cases

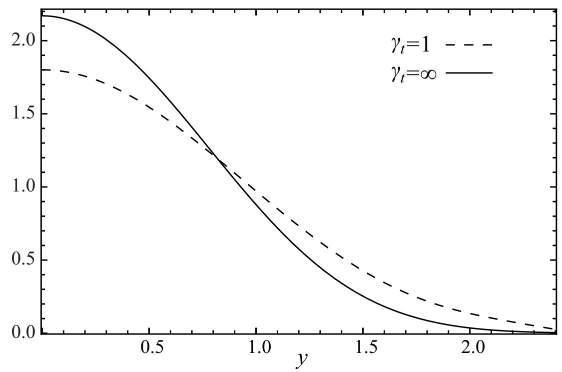
Wedge	Stationary	Flat
		
Simplest	Non-eq stat. state	Standard for growth
GUE TW dist	$\boldsymbol{F_0}$ (BR dist)	GOE TW dist
$\boldsymbol{e^{h(x,0)} = \delta(x)}$	$\boldsymbol{h(x,0) = B(x)}$	$\boldsymbol{h(x,0) = 0}$

Stationary two point correlation

Imamura TS (2012)

$$\langle \partial_x h(x,t)\partial_x h(0,0)\rangle = \frac{1}{2}(2t)^{-2/3} g_t''(x/(2t)^{2/3})$$





The solid curve is the scaling limit $g''(y)$.

For ASEP corresponds to 2-pt correlation $\langle n_j(t)n_0(0)\rangle$.

Short summary of Part 1

- ASEP is a microscopic model with Markov dynamics and has a single conserved quantity.
- Hydrodynamics is well-known (Burgers equation).
- Fluctuations can be studied exactly (integrable probability).
- Adding noise to hydrodynamics, one gets noisy Burgers equation ($\sim$ KPZ equation).
- This can also be studied exactly with the same universal behaviors with ASEP.

2. Nonlinear fluctuating hydrodynamics

Fermi-Pasta-Ulam chain

$$\boldsymbol{H = \sum_j \left(\frac{p_j^2}{2} + V(x_{j+1} - x_j) \right)}$$

with

$$\boldsymbol{V(x) = \frac{1}{2}x^2 + \frac{\alpha}{3}x^3 + \frac{\beta}{4}x^4}$$

- Microscopic model with Hamiltonian dynamics.
- FPU tried to see thermalization numerically, but recurrence seemed to occur. ($\to$ Chaos, Soliton, ...)
- It is still difficult to study large time behaviors of the model, but some aspects may be understood by using a connection to KPZ through nonlinear fluctuating hydrodynamics.

Nonlinear fluctuating hydrodynamics

A conjectural theory for 1D multi-component systems which predicts that the distributions and correlations of "normal modes" are described by the ones of the single-component KPZ equation.

van Beijreren 2011, Spohn 2013-

For an anharmonic chain, there are three conserved fields.

$$r_j = x_{j+1} - x_j, \;\; p_j, \;\; e_j = \frac{p_j^2}{2} + V(r_j)$$

Equations of motion

$$\begin{aligned}
\dot{r}_j &= p_{j+1} - p_j \\
\dot{p}_j &= V'(r_j) - V'(r_{j-1}) \\
\dot{e}_j &= p_{j+1}V'(r_j) - p_j V'(r_{j-1})
\end{aligned}$$

This can be summarized as the continuity equation

$$\frac{d}{dt}\vec{G}(j,t) + \vec{J}(j+1,t) - \vec{J}(j,t) = 0$$

where

$$\vec{G} = (r_j, p_j, e_j) \quad \vec{J} = (-p_j, -V'(r_{j-1}), -p_j V'(r_{j-1}))$$

Hydrodynamics: Euler equation (like Burgers for ASEP)

$$\frac{\partial}{\partial t}\vec{g} + \frac{\partial}{\partial x}\vec{j} = 0$$

For taking into account the fluctuations effectively, we add noise (η) and a diffusion term to get a SPDE

$$\frac{\partial}{\partial t}\vec{g} + \frac{\partial}{\partial x}(\vec{j} + \partial_x D\vec{g} + B\vec{\eta}) = 0$$

Remark: This "derivation" is heuristic, not at all rigorous.

Expanding around equilibrium up to the 2nd order, we find

$$\frac{\partial}{\partial t}\vec{u} + \frac{\partial}{\partial x}\left(A\vec{u} + \frac{1}{2}\langle\vec{u}, H\vec{u}\rangle + \partial_x D\vec{u} + B\vec{\eta}\right) = 0$$

Diagonalizing A as $RAR^{-1} = \mathrm{diag}(c_1, c_0, c_{-1})$ and setting $\vec{\phi} = R\vec{u}$ (normal modes), we get

$$\frac{\partial}{\partial t}\phi_\alpha + \frac{\partial}{\partial x}\left(c_\alpha\phi_\alpha + \frac{1}{2}\langle\vec{\phi}, G^\alpha\vec{\phi}\rangle + \partial_x(D\vec{\phi})_\alpha + (B\vec{\eta})_\alpha\right) = 0$$

This is multi-component KPZ eq (well-definedness by Funaki-Hoshino). Assuming main contributions from the nonlinear term come from diagonal terms, the equation for a component is noisy-Burgers equation.

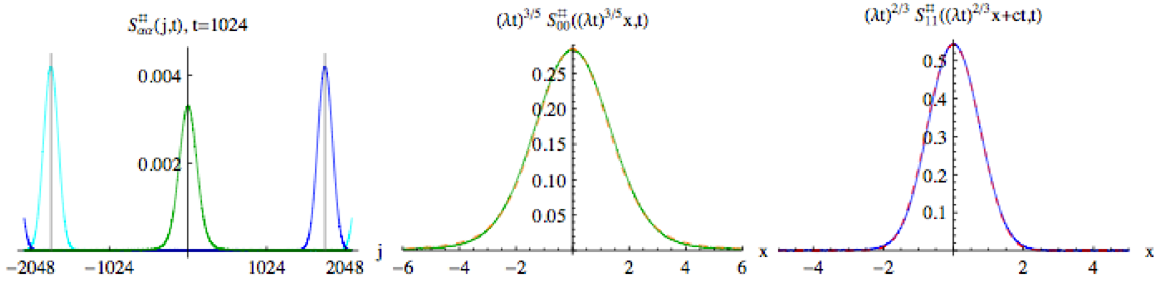
$\Rightarrow$ Correlations and distributions for the normal modes are expected to be described by the ones of the KPZ equation.

Simulations and problem

For an anharmonic chain, $c_{\pm} = \pm c, c_0 = 0$, corresponding to two sound modes and one heat mode. The correlation of each sound mode seems to be given by the stationary KPZ 2pt function.

MD simulations for shoulder potential (Mendl Spohn)

$$V(x) = \infty \ (0 < x < \tfrac{1}{2}), 1(\tfrac{1}{2} < x < 1), \ 0(x > 1)$$



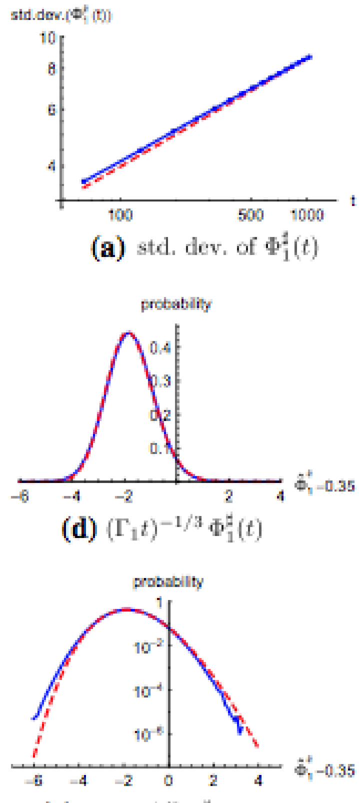
(a) overview (b) heat, $\lambda = 1.624$ (c) sound, $\lambda = 1.442$

Problem 1: Can we prove them (or Show them analytically) ?

Universal KPZ distributions?

Mendl Spohn

- By considering the integrated current for normal modes, one can also observe $\boldsymbol{F_0}$.
- For step type initial condition in which the macro paramters (temperature, pressure) change at the origin, one observes GUE TW.



(a) std. dev. of $\Phi_1^\sharp(t)$

(d) $(\Gamma_1 t)^{-1/3}\,\Phi_1^\sharp(t)$

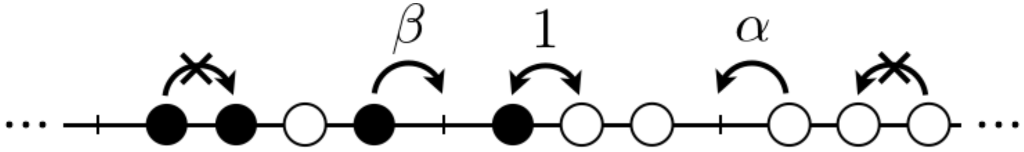
(g) $(\Gamma_1 t)^{-1/3}\,\Phi_1^\sharp(t)$

3. NLFHD for stochastic models

- NLFHD theory applies to more general multi-component systems with more than one conserved quantities.
- Stochastic systems should be easier to treat.
- There are various interesting problems from points of view of interacting particle systems and integrable probability.
- The case where the "sound modes" are linear are easier and there have been several results (fractional diffusion for momentum exchange model by Olla et al etc). The case of KPZ fluctuations have been less studied.

Two species ASEP

AHR model (1998 Arndt-Heinzel-Rittenberg)



●: $+$ particle
○: $-$ particle

- The model has two conserved quantities (numbers of $\pm$ particles). NLFHD can be applied.

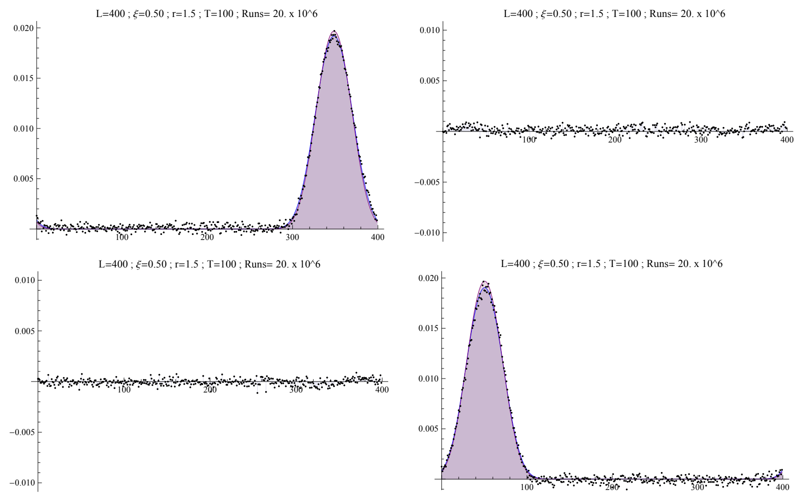
- We will focus on the case where $\boldsymbol{q = 0}$ and $\boldsymbol{\alpha + \beta = 1}$, for which the stationary measure is factorized. (Mostly $\boldsymbol{\alpha = \beta = \frac{1}{2}}$ below.)

- The integrability was shown by Cantini 2008.

  There are multi-species ASEP with $\boldsymbol{U_q(sl_n)}$ symmetry but AHR is different.

Monte Carlo simulation for AHR model

For AHR model, stationary KPZ 2pt function had been observed in MC simulations (Ferrari TS Spohn 2014).

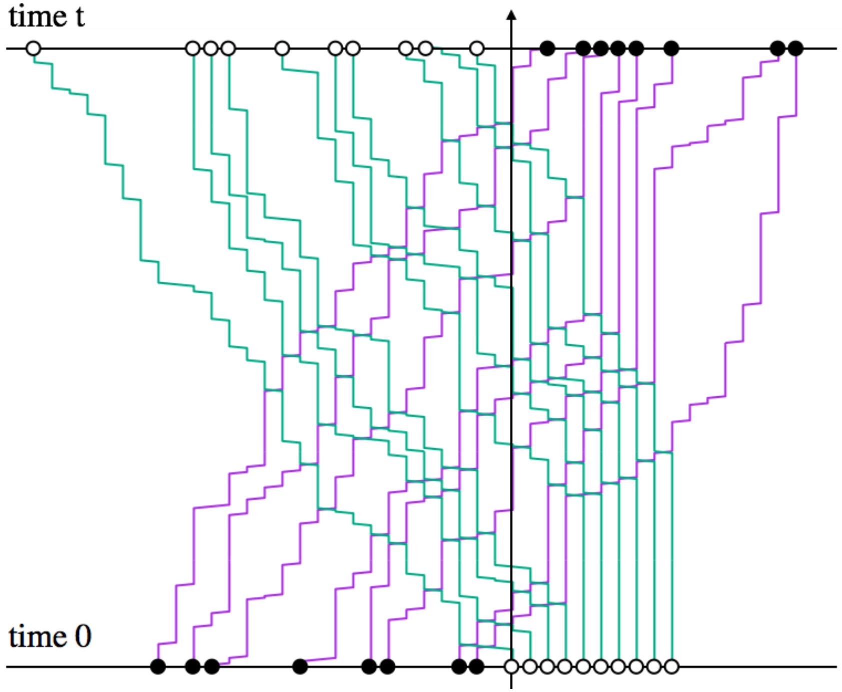


For step i.c. GUE TW was observed (Mendl Spohn)

Problem 2: Can we prove these?

$\rho - 1$ Step i.c.

Infinite $+$ particles ($\bullet$) with density ρ on the left and infinite $-$ particles ($\circ$) packed on the right.



Hydrodynamics

Fritz Toth 2004

Macroscopic densities of $\pm$ particles, $\boldsymbol{u(t,x) = (\rho_+, \rho_-)}$, satisfy

$$\frac{\partial \boldsymbol{u}(t,x)}{\partial t} + \frac{\partial \mathbf{j}(\boldsymbol{u}(t,x))}{\partial x} = \mathbf{0}$$

where $\mathbf{j}_\pm(\boldsymbol{u})$ represent macroscopic current of $\pm$ particle,

$$\mathbf{j}_+(\boldsymbol{u}) = \boldsymbol{\rho}_+(1 - \boldsymbol{\rho}_+ - \boldsymbol{\rho}_-) + 2\boldsymbol{\rho}_+\boldsymbol{\rho}_-$$
$$\mathbf{j}_-(\boldsymbol{u}) = -(1 - \boldsymbol{\rho}_+ - \boldsymbol{\rho}_-)\boldsymbol{\rho}_- - 2\boldsymbol{\rho}_+\boldsymbol{\rho}_-$$

This set of coupled equations can be solved.

Normal modes for $\rho - 1$ step i.c.

The NLFHD predicts

$$\lim_{t\to\infty} P_{\infty,\infty}[s_-(n,m,\rho,t) \le s_-] = F_2(s_-)$$
$$\lim_{t\to\infty} P_{\infty,\infty}[s_+(n,m,\rho,t) \le s_+] = F_G(s_+)$$

where n, m are the numbers of $\pm$ particles which passed the origin up to time t and scaled variables $s_\pm(n,m,\rho,t)$ are given by

$$s_-(n,m,\rho,t) = \frac{(1+\rho)\cdot n - (3-\rho)\cdot m + (1-\rho)(1-\frac{(1-\rho)^2}{4})t}{(3/16)^{1/3}(1-\rho)(3-\rho)^{2/3}(1+\rho)^{2/3}t^{1/3}}$$

$$s_+(n,m,\rho,t) = \frac{-2(2-\rho)\cdot n + 2\rho\cdot m + 2(2-\rho)(1-\rho)\rho t}{3(1-\rho)^{3/2}\sqrt{\rho(2-\rho)}t^{1/2}}$$

4. Exact confirmation of NLFHD for ρ-1 AHR model

The AHR (with $q = 0$) is exactly solvable.

- By using Bethe ansatz, one can find eigenfunctions for the generator of the AHR model.
- When $\alpha + \beta = 1$, the stationary measure is Bernoulli. We focus on this case (we will set $\alpha = \beta = 1/2$ later).

Transition probability (Green's function)

Let $x_i^{(0)}, y_j^{(0)}$ be initial positions of $+$ and $-$ particles and x_i, y_j final positions at time t. Using the Bethe ansatz eigenfunctions, for certain special cases, one can write down the Green's function (probability of x_i, y_j at t given $x_i^{(0)}, y_j^{(0)}$ at $t = 0$) as a multiple integral.

Assume $x_1^{(0)} < \ldots < x_N^{(0)} < y_1^{(0)} < \ldots < y_M^{(0)}$.

The Green's function for the case (the order has not changed)

$$x_1 < \ldots < x_N < y_1 < \ldots < y_M,$$

is given by, with $\boldsymbol{\Lambda = \beta \sum_{i=1}^{N}(z_i^{-1} - 1) + \alpha \sum_{i=1}^{M}(w_i^{-1} - 1)}$,

$$\begin{aligned}
&G(\{x_j - x_j^{(0)}\}, \{y_k - y_k^{(0)}\}, t) \\
&= \oint \prod_{j=1}^{N} \frac{\mathrm{d}z_j}{2\pi \mathrm{i}} \prod_{k=1}^{M} \frac{\mathrm{d}w_k}{2\pi \mathrm{i}}\ \mathrm{e}^{\Lambda t} \\
&\times \sum_{\pi \in S_N} \mathrm{sign}(\pi) \prod_{j=1}^{N} \left(\frac{z_j - 1}{z_{\pi_j} - 1} \right)^{j-1} z_{\pi_j}^{x_j} z_j^{-x_j^{(0)}-1} \\
&\times \sum_{\rho \in S_M} \mathrm{sign}(\rho) \prod_{k=1}^{M} \left(\frac{w_k - 1}{w_{\rho_k} - 1} \right)^{M-k} w_{\rho_k}^{-y_k} w_k^{y_k^{(0)}-1},
\end{aligned}$$

where all contour integrals are around the origin.

$+- \;\to\; -+$ boundary conditions

For the ordering (complete exchange of $+$ and $-$ particles)

$$y_1 < \ldots < y_M < x_1 < \ldots < x_N,$$

$$\begin{aligned}
&G(\{x_j - x_j^{(0)}\}, \{y_k - y_k^{(0)}\}, t) \\
&= \oint \prod_{j=1}^{N} \frac{\mathrm{d}z_j}{2\pi\mathrm{i}} \prod_{k=1}^{M} \frac{\mathrm{d}w_k}{2\pi\mathrm{i}}\, \mathrm{e}^{\Lambda t} \prod_{k=1}^{M} \prod_{j=1}^{N} \frac{1}{\beta z_j + \alpha w_k} \\
&\times \sum_{\pi \in S_N} \mathrm{sign}(\pi) \prod_{j=1}^{N} \left(\frac{z_j - 1}{z_{\pi_j} - 1} \right)^{j-1} z_{\pi_j}^{x_j} z_j^{-x_j^{(0)}-1} \\
&\times \sum_{\rho \in S_M} \mathrm{sign}(\rho) \prod_{k=1}^{M} \left(\frac{w_k - 1}{w_{\rho_k} - 1} \right)^{M-k} w_{\rho_k}^{-y_k} w_k^{y_k^{(0)}-1}
\end{aligned}$$

Multiple-integral formula for current distribution

A step i.c. in which there are N $+$ particles on the left with density ρ and M $-$ particles are packed on the right.

When $\alpha+\beta=1$, for the currents $N_\pm(t)$ at the origin,

$$P_{N,M}(N_+(t)=N, N_-(t)=M) = \frac{1}{N!M!}\oint \prod_{j=1}^{N}\frac{\mathrm{d}z_j}{2\pi\mathrm{i}}\prod_{k=1}^{M}\frac{\mathrm{d}w_k}{2\pi\mathrm{i}}\,\mathrm{e}^{\Lambda t}$$

$$\frac{\rho^N \prod_{1\le i<j\le N}(z_i-z_j)^2 \prod_{1\le k<l\le M}(w_l-w_k)^2}{\prod_{j=1}^{n}(z_j-1)^N(1-(1-\rho)z_j)\prod_{k=1}^{M}(w_k-1)^M\prod_{j=1}^{N}\prod_{k=1}^{M}\left(\alpha z_j+\beta w_k\right)}$$

with $\Lambda=\sum_{j=1}^{N}\beta(1/z_j-1)+\sum_{k=1}^{M}\alpha(1/w_k-1)$.

NLFHD prediction for finite particles case

- The original NLFHD is formulated for infinite systems.

$$P_{\infty,\infty}[N_+(t)=n, N_-(t)=m] \simeq F_G'(s_+)F_2'(s_-)$$

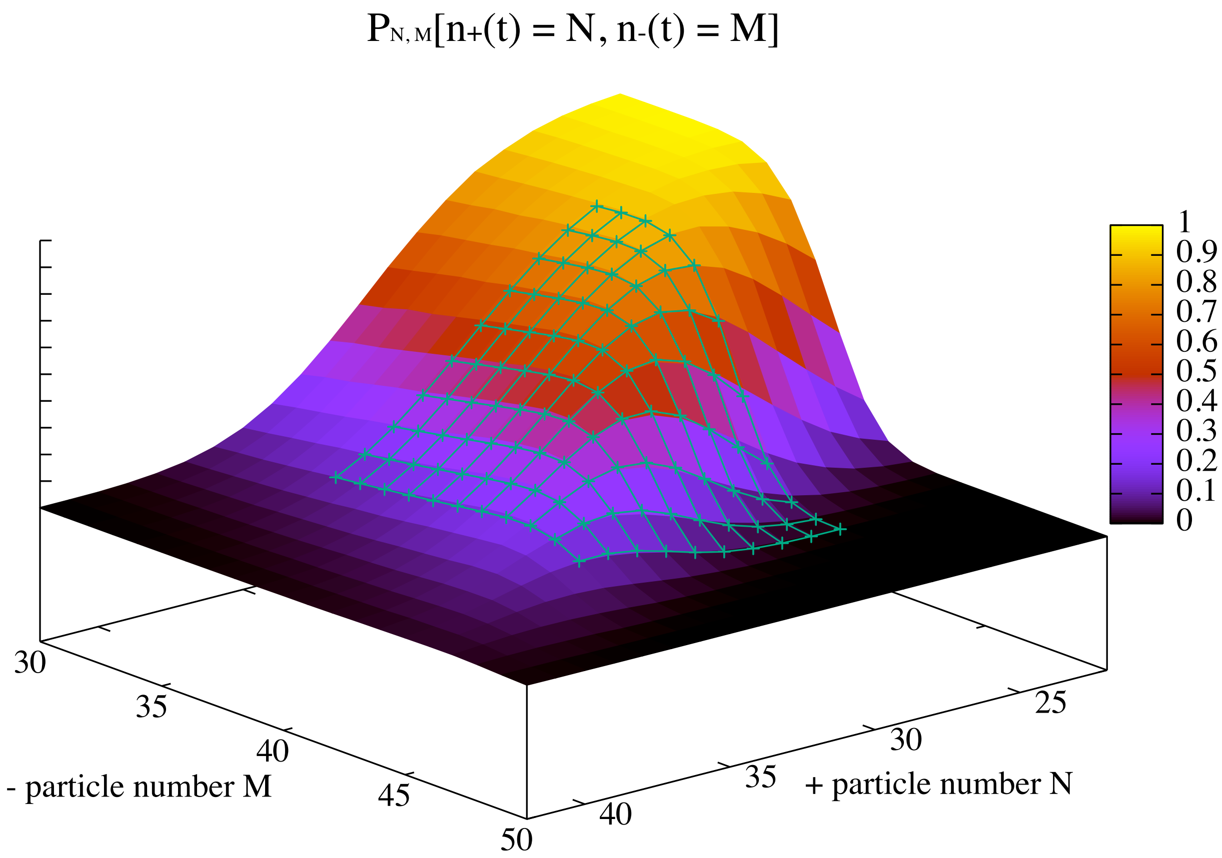
- Our formula is for finite number of particles. We can formulate a generalization of NLFHD prediction for finite case.

  Conjecture (Theorem (2021 Chen, de Gier, Hiki, TS, Usui))

$$\lim_{t\to\infty} P_{N,M}[N_+(t)=N, N_-(t)=M] = F_G(s_+)F_2(s_-)$$

  This may look very similar to the original conjecture but is in fact a very nontrivial generalization. (Note the difference of $P_*[N_+(t)=N, N_-(t)=M]$ as $t\to\infty$.)

Confirmation of the conjecture by simulation



Analytic confirmation by the multiple integral formula

In the multiple integral formula, we take the simple pole at $z_j = 1/(1-\rho)$ and find

$$P_{N,M}[N_+(t) = N, N_-(t) = M] = I_1 + J \times I_2$$

where

$$I_1 = \frac{1}{M!} \int \prod_{k=1}^{M} \frac{dw_k}{2\pi i} \frac{e^{\Lambda_{0,M} t} \prod_{1 \le k < l \le M} (w_l - w_k)^2}{\prod_{k=1}^{M} (w_k - 1)^M \prod_{k=1}^{M} (\frac{1}{2}(1 + w_k))^N}$$

$$J = \frac{\rho^{N-1}}{(1-\rho)^N} \left(\frac{2(1-\rho)}{2-\rho}\right)^M \frac{e^{-\rho t/2}}{(N-1)!} \int_1 \prod_{j=1}^{N-1} \frac{dz_j}{2\pi i}$$

$$\times\, e^{\Lambda_{N-1,0} t} \frac{\prod_{1\le i<j\le N-1}(z_i - z_j)^2 \prod_{j=1}^{N-1}(1-(1-\rho)/z_j)}{\prod_{j=1}^{N-1}(z_j-1)^N \prod_{j=1}^{N-1}(\frac{1}{2}(1+z_j))^M},$$

$$I_2 = \frac{1}{M!} \int \prod_{k=1}^{M} \frac{dw_k}{2\pi} e^{\Lambda_{0,M} t} \frac{\prod_{1\le k<l\le M}(w_l - w_k)^2}{\prod_{k=1}^{M}(w_k-1)^M}$$

$$\times\, \frac{\prod_{j=1}^{N-1}(1+z_j)^M (\frac{1}{1-\rho}+1)^M}{\prod_{k=1}^{M}(\prod_{j=1}^{N-1}(z_j+w_k)(\frac{1}{1-\rho}+w_k))}.$$

We can study asymptotics of the integrals to show

$I_1, I_2 \simeq F_2(s_-), J \simeq F_G(s_+) - 1$.

The first analytic confirmation of the KPZ prediction of NLFHD!

A key observation for the proof

By a standard procedure to rewrite a mutliple integral into a Fredholm determinant,

$$I_2 = \det(1 - K_{z_j})_{\ell_2(\mathbb{N})}$$

where the kernel is given by, with $\rho' = 1 - \rho$,

$$K_{z_j}(x,y) = \oint_1 \frac{\mathrm{d}z}{2\pi i} \frac{z+\rho'}{z^{x+1}} \left(\frac{z}{1-z}\right)^m \prod_{j=1}^{n-1} \frac{1+z_j z}{z} \mathrm{e}^{-zt/2} \frac{1}{w-z}$$

$$\times \oint_{0,-\rho',\{-1/z_j\}_{j=1}^{n-1}} \frac{\mathrm{d}w}{2\pi i} \frac{w^y}{w+\rho'} \left(\frac{1-w}{w}\right)^m \prod_{j=1}^{n-1} \frac{w\mathrm{e}^{wt/2}}{1+z_j w}.$$

This still depends on $\{z_j\}$'s but can be written as a rank-one perturbation, inside the mutliple integral over $\{z_j\}$'s.

$$K_{z_j}(x_i, x_j) = K_1(x_i, x_j) - \left[\sum_{p=1}^{n-1} \prod_{q=1}^{p} (z_q - 1) A_p(x_i)\right] B(x_j)$$

$K_1(x, y)$ is the kernel which is obtained by setting all of z_js to 1 in $K_{z_j}(x, y)$ and $A_j(x)$ and $B(y)$ are of the form,

$$A_j(x) = \oint_1 \frac{\mathrm{d}z}{2\pi i} f(x; z) \left(\frac{1+z}{z}\right)^{n-j} \frac{1}{1+z}$$

$$B(y) = \oint_{0, -\rho', -1} \frac{\mathrm{d}w}{2\pi i} g(y; w) \left(\frac{w}{1+w}\right)^{n-1} \frac{1}{1+w}$$

One can show that the contribution from the perturbation part is small for large t. One can also estimate how the error decays.

Summary

- Nonlinear fluctuating hydrodynamics is a conjectural physical theory which can predict correlation and fluctuations of multi-component systems, including Hamiltonian dynamics.

- The conjectures have been tested in Monte Carlo simulations with reasonably good agreement.

  Theoretical and mathematical understanding is unsatisfactory.

- The conjectures apply also to stochastic models, which seems more tractable.

- We have given a confirmation of the conjecture for a two species exclusion process by exact calculations using techniques of integrable probability.

Open problems

Stochastic interacting particle systems

- Establishing hydrodynamics for multi-component systems.
- Deriving multi-component KPZ equations for multi-component systems.
- Establishing universal behaviors,....

Integrable probability

- Other initial conditions, boundary conditions, quantities for AHR model
- More general parameters for AHR model
- Other multi-component models,...