Stochastic homogenization of some nonconvex Hamilton-Jacobi equations

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PDEs and Probability Theory
-beyond boundaries-

Main goals

- Obtain homogenization results for Hamilton-Jacobi equations

$$\begin{cases} u^\epsilon_t + H(Du^\epsilon) - V\left(\frac{x}{\epsilon}\right) = 0 & \text{in } \mathbb{R}^n \times (0,\infty), \\ u^\epsilon(x,0) = g(x) & \text{on } \mathbb{R}^n. \end{cases} \tag{1}$$

  In practice, $\epsilon > 0$ is a fixed length scale, which is quite small.
- **Qualitative properties, representation formulas** of the effective Hamiltonian in homogenization theory (in both periodic and general stationary ergodic settings).

Homogenization theory aims at studying macroscopic behavior of PDEs which typically have high oscillations in the space (or time-space) variables. Basic problems include

(I) well-posedness: obtaining the existence of limiting effective equations as $\epsilon \to 0$;

(II) understanding finer properties of the limiting process and the effective equation.

PDEs are usually set in self-averaging (periodic, almost periodic or random) environments. In the periodic setting, (I) is quite well established for some nonlinear PDEs (e.g., first-order and second-order HJ equations, fully nonlinear elliptic equations). Not much in the random setting because of the lack of compactness.

Very little is known about (II) because of the nonlinear nature in these equations.

Periodic homogenization of Hamilton-Jacobi equations

For each $\epsilon > 0$, let $u^\epsilon \in C(\mathbb{R}^n \times [0,\infty))$ be the viscosity solution to

$$\begin{cases} u_t^\epsilon + H(Du^\epsilon) - V\left(\frac{x}{\epsilon}\right) = 0 & \text{in } \mathbb{R}^n \times (0,\infty), \\ u^\epsilon(x,0) = g(x) & \text{on } \mathbb{R}^n. \end{cases}$$

Here the Hamiltonian is of separable form with $H \in C(\mathbb{R}^n)$, which is coercive, and $V \in C(\mathbb{R}^n)$, which is $\mathbb{Z}^n$-periodic.

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It was known (Lions-Papanicolaou-Varadhan, 1987), that u^ϵ, as $\epsilon \to 0$, converges locally uniformly to u, the solution of the effective equation,

$$\begin{cases} u_t + \overline{H}(Du) = 0 & \text{in } \mathbb{R}^n \times (0,\infty), \\ u(x,0) = g(x) & \text{on } \mathbb{R}^n. \end{cases} \tag{2}$$

$\overline{H} : \mathbb{R}^n \to \mathbb{R}$ is called the effective Hamiltonian.

Heuristic derivation

Search for an ansatz for u, and a correct one is

$$u^\epsilon(x,t) = u(x,t) + \epsilon v(\frac{x}{\epsilon}) + \cdots = u(x,t) + \epsilon v(y) + \cdots ,$$

where x is the macro variable, and $y = \frac{x}{\epsilon}$ is the micro variable.
Plug this into the PDE to get

$$u_t(x,t) + H(Du(x,t) + Dv(y)) - V(y) = 0.$$

Assume now that x and y are unrelated. Fix (x,t) and think of the above PDE as an equation in y. Let $p = Du(x,t) \in \mathbb{R}^n$, and $c = -u_t(x,t) \in \mathbb{R}$,

$$H(p + Dv(y)) - V(y) = c.$$

Effective Hamiltonian $\overline{H}$

For any $p \in \mathbb{R}^n$, there exists a UNIQUE number $\overline{H}(p)$ such that

$$H(p + Dv) - V(y) = \overline{H}(p) \quad \text{in } \mathbb{R}^n, \tag{3}$$

has a periodic solution (corrector) $v = v(y)$.

Evans: perturbed test function method.

However, in general, deep properties of $\overline{H}$ are not known so much, especially in the nonconvex setting.

In the stationary ergodic setting, we don't have compactness, and (3) might not have sublinear solutions.

Stationary ergodic setting

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. Suppose that $\{\tau_y\}_{y\in\mathbb{R}^n}$ is a measure-preserving translation group action of $\mathbb{R}^n$ on Ω satisfying

(1) (**Semi-group property**)

$$\tau_x \circ \tau_y = \tau_{x+y} \quad \text{for all } x, y \in \mathbb{R}^n.$$

(2) (**Ergodicity**) For any $E \in \mathcal{F}$,

$$\tau_x(E) = E \quad \text{for all } x \in \mathbb{R}^n \quad \Rightarrow \quad \mathbb{P}(E) = 0 \quad \text{or} \quad \mathbb{P}(E) = 1.$$

The potential $V(x,\omega) : \mathbb{R}^n \times \Omega \to \mathbb{R}$ is stationary, bounded and uniformly continuous. More precisely, $V(x+y,\omega) = V(x,\tau_y\omega)$ for all $x, y \in \mathbb{R}^n$ and $\omega \in \Omega$, $\operatorname{ess\,sup}_\Omega |V(0,\omega)| < +\infty$ and

$$|V(x,\omega) - V(y,\omega)| \le c(|x-y|) \quad \text{for all } x, y \in \mathbb{R}^n \text{ and } \omega \in \Omega,$$

for some function $c : [0,\infty) \to [0,\infty)$ satisfying $\lim_{r\to 0} c(r) = 0$.

Nonexistence of sublinear correctors

An 1-D example: $H(p) - V(y) = |p| - (2 - \cos y - \cos(\sqrt{2}y))$ by Lions-Souganidis. At $p = 0$, we have $\overline{H}(0) = 0$, and corrector problem is

$$|v'(y)| = V(y) = 2 - \cos y - \cos(\sqrt{2}y), \quad y \in \mathbb{R}.$$

Note: $V \geq 0$, and $V(y) = 0$ iff $y = 0$. Geometrically, the graph of v cannot have corners from below at points $y \neq 0$. So, there exists $y_0 \in \mathbb{R}$ such that $v'(y)$ doesn't change sign for $y > y_0$. Let's consider the case $v'(y) = 2 - \cos y - \cos(\sqrt{2}y)$ for $y > y_0$. Then,

$$\frac{v(y)}{|y|} \geq \frac{1}{|y|}\left(\int_{y_0}^{y} V(s)\,ds - |v(y_0)|\right) \geq 2 - \frac{C}{|y|},$$

which means that v is not sublinear.

Homogenization in the almost periodic setting: Ishii.

Stochastic homogenization of first-order HJ eqns

- Convex/quasiconvex case: Souganidis (1999) and Rezakhanlou-Tarver (2000). Davini-Siconolfi (2009), Armstrong-Souganidis (2013).

• **Major open problem.** Stochastic homogenization of nonconvex cases?
- Specific n-D case: $H(p) = (|p|^2 - 1)^2$ (Armstrong -T.-Yu (2013)).
- One dimension: Armstrong -T.-Yu (2014) for general separable case $H(p) - V(x)$, Gao (2015) for general non-separable case.
- Counterexample: If H has strict saddle point, then there exists V such that homogenization does not hold (Ziliotto (2016), Feldman-Souganidis (2016)).
- In i.i.d setting: Armstrong-Cardaliaguet (2015), Feldman-Souganidis (2016) for k-positively homogeneous H.
- Some general n-D cases: Qian-T.-Yu.

Some ideas in the convex case 1

Replace the corrector problem by the metric problem: For $z \in \mathbb{R}^n$ fixed,

$$\begin{cases} H(Dv(x)) - V(x,\omega) = \mu & \text{in } \mathbb{R}^n \setminus \{z\}, \\ v(z) = 0. \end{cases}$$

Here, $\mu \in \mathbb{R}$ is a parameter. Let $m_\mu(x,z)$ be the maximal solution to the above, that is,

$$m_\mu(x,z,\omega) = \sup\{v(x) \,:\, v \text{ is a subsolution to the above}\}.$$

Then, m_μ has the subadditive property

$$m_\mu(x,z,\omega) \le m_\mu(x,y,\omega) + m_\mu(y,z,\omega).$$

By the subadditive ergodic theorem and some further deductions

$$\lim_{t\to\infty} \frac{1}{t} m_\mu(tx,0,\omega) = \overline{m}_\mu(x).$$

Some ideas in the convex case 2

How do we relate this large time average to homogenization? For $\epsilon > 0$, let $m^\epsilon_\mu(x,\omega) = \epsilon m_\mu(\frac{x}{\epsilon}, 0, \omega)$. Then, m^ϵ_μ solves

$$\begin{cases} H(Dm^\epsilon_\mu) - V(\frac{x}{\epsilon},\omega) = \mu \quad \text{in } \mathbb{R}^n \setminus \{0\}, \\ m^\epsilon_\mu(0,\omega) = 0. \end{cases}$$

We already got $m^\epsilon_\mu(x,\omega) \to \overline{m}_\mu(x)$, and we should have

$$\begin{cases} \overline{H}(D\overline{m}_\mu) = \mu \quad \text{in } \mathbb{R}^n \setminus \{0\}, \\ \overline{m}_\mu(0) = 0. \end{cases}$$

And this helps to identify $\overline{H}$ (Armstrong-Souganidis, Armstrong-T.)

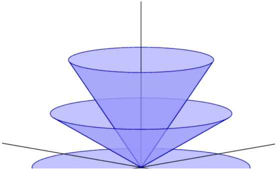
$$\overline{H}(p) = \inf\{\mu : \overline{m}_\mu(y) \geq p \cdot y \text{ for all } y \in \mathbb{R}^n\}.$$

Some ideas in the convex case 3

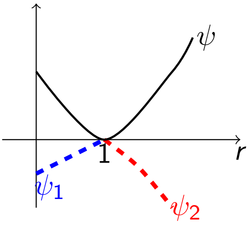
Representation formula of m_μ, which is related to first passage percolation.

$$m_\mu(x,y,\omega) = \inf\left\{ \int_0^t \left(L(\dot\gamma(s)) + V(\gamma(s),\omega) + \mu\right)\, ds \ : \ t>0, \gamma(0)=y, \gamma(t)=x \right\}.$$

See [T., Chapter 2] for more. Shapes of $\{\overline{m}_\mu\}_\mu$ and identification of $\overline{H}$.



New results - Simplest case



Let $H(p) = \psi(|p|)$, $H_1(p) = \psi_1(|p|)$, and $H_2(p) = \psi_2(|p|)$. Let $\overline{H}, \overline{H}_1, \overline{H}_2$ be the effective Hamiltonians of $H - V$, $H_1 - V$, $H_2 - V$, respectively.

Theorem (Qian - T. - Yu)

Assume the above, and $V \in C(\mathbb{T}^n)$ *with* $\min V = 0$. *Then*

$$\overline{H} = \max\left\{\overline{H}_1, \overline{H}_2, 0\right\}.$$

Important consequences

- $\overline{H}$ is even as $\overline{H}_1, \overline{H}_2$ are even.
- In case that $\min V = 0$ and $\max V \geq \psi(0)$, then $\overline{H}_2 \leq 0$. We then get

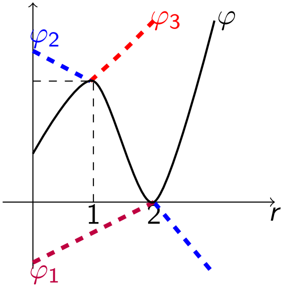
$$\overline{H} = \max\left\{\overline{H}_1, 0\right\},$$

and thus, $\overline{H}$ is quasiconvex. Strong V makes $\overline{H}$ better.

- In case that $\min V = 0$ and $\max V \geq \psi(0)$. Let $K : \mathbb{R}^n \to \mathbb{R}$ such that $K(p) = H(p) = \psi(|p|)$ for $|p| \geq 1$, and $0 \leq K(p) \leq H(p)$ for $|p| \leq 1$. Let $\overline{K}$ be the effective Hamiltonian corresponding to $K - V$. Then we always have

$$\overline{K} = \overline{H}.$$

New results - A bit more general case

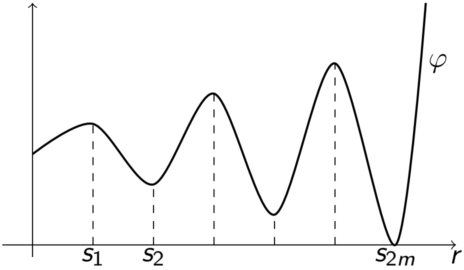


Theorem (Qian-T.-Yu)

Let $H(p) = \varphi(|p|)$ *and* $V \in C(\mathbb{T}^n)$. *Then,*

$$\overline{H} = \max\left\{0, \overline{H}_1, \min\left\{\overline{H}_2, \overline{H}_3, \varphi(1) - \max V\right\}\right\}.$$

New results - Most general case



Theorem (Qian-T.-Yu)

Let $H(p) = \varphi(|p|)$. Then we have a representation formula

$$\overline{H} = \max\ \min\ \max\ \min \ldots$$

Related results: Armstrong-T.-Yu (2013, 2014), Gao (2015, 2018).

Application: Random homogenization - New results

Theorem (Qian-T.-Yu)

Stochastic homogenization holds for all H mentioned in Theorems 1–3. Moreover, representation formulas also hold true.

Key philosophy. Knowledge on $\overline{H}$ (in periodic setting) helps to recover same formulas in random setting, and overcome the lack of compactness.

Counterexamples of Ziliotto (2016), Feldman-Souganidis (2016) require V to have small oscillation to see the local structure of the saddle points. When the oscillation of V is large enough, we do not see such local structure and hence we still have the averaging effect in certain cases.

More or less, the whole set of new developments gives rather clear answers to the open question in the general stationary ergodic setting.

• It is clear that $\overline{H} \geq \overline{H}_i$ as $H \geq H_i$ for $i = 1, 2$. Also, $\overline{H} \geq 0$ and $\overline{H}(p) = 0$ for $|p| = 1$. Therefore, $\overline{H} \geq \max\{\overline{H}_1, \overline{H}_2, 0\}$.

• Pick $p \in \mathbb{R}^n$ such that $\overline{H}_1(p) \geq \max\{\overline{H}_2(p), 0\}$. We show

$$\overline{H}_1(p) \geq \overline{H}(p).$$

As $\overline{H}_1$ is even, $\overline{H}_1(-p) = \overline{H}_1(p)$. Let $v(y, -p)$ be a solution to

$$H_1(-p + Dv(y, -p)) - V(y) = \overline{H}_1(-p) = \overline{H}_1(p) \quad \text{in } \mathbb{T}^n.$$

Let $w(y) = -v(y, -p)$. If $q \in D^+ w(y)$, then $-q \in D^- v(y, -p)$ and

$$0 \leq \overline{H}_1(p) = H_1(-p - q) - V(y) = H_1(p + q) - V(y) = H(p + q) - V(y).$$

Thus, w is a subsolution, and the conclusion follows.

Some ideas in the proof of Theorem 1

• Pick $p \in \mathbb{R}^n$ such that $\overline{H}_2(p) \geq \max\{\overline{H}_1(p), 0\}$. This is similar to the above, but we just need to choose $w(y) = v(y, p)$.

• **Gluing step.** Assume $\max\{\overline{H}_1(p), \overline{H}_2(p)\} < 0$. We show $\overline{H}(p) = 0$. For $\sigma \in [0,1]$, let $\overline{H}^\sigma, \overline{H}_i^\sigma$ be effective Hamiltonians corresponding to $H(p) - \sigma V(y)$, $H_i(p) - \sigma V(y)$, respectively. It is clear that

$$0 \leq \overline{H} = \overline{H}^1 \leq \overline{H}^\sigma \quad \text{for all } \sigma \in [0,1].$$

WLOG, assume $|p| > 1$. As $H_1(p) = \overline{H}_1^0(p) > 0$ and $\overline{H}_1(p) = \overline{H}_1^1(p) < 0$, we can find $s \in (0,1)$ such that $\overline{H}_1^s(p) = 0$. It is clear that $\overline{H}_2^s(p) \leq H_2(p) < 0$ there. Thus,

$$\max\{\overline{H}_1^s(p), \overline{H}_2^s(p)\} = 0 \Rightarrow \overline{H}^s(p) = 0 \Rightarrow \overline{H}(p) = 0.$$

Conclusion

- Still many important questions to be studied in qualitative homogenization theory.
- Quantitative results: Optimal rates of convergence? Largely open so far in the random setting.
  In the periodic setting: Mitake-T.-Yu.

THANK YOU