Quasi-Invariance of Gaussian Measures Transported by the Cubic NLS with Third-Order Dispersion on $\mathbf{T}$

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1 Introduction

- Cubic NLS with 3rd order dispersion.

$$\partial_t u = -i(i\partial_x^3 + \beta\partial_x^2)u - i|u|^2 u, \tag{1}$$

$$t \in \mathbf{R}, \quad x \in \mathbf{T} = \mathbf{R}/2\pi\mathbf{Z},$$

$$u(0, x) = u_0(x), \quad x \in \mathbf{T}, \tag{2}$$

where $u : \mathbf{R} \times \mathbf{T} \to \mathbf{C}$ and β is the dispersion constant.
Assume throughout this talk that

$$2\beta/3 \notin \mathbf{Z}\backslash\{0\} \tag{NR}$$

- Conservation Laws

  Mass Conservation: $\|u(t)\|_{L^2} = \|u_0\|_{L^2}$,

  Energy Conservation:

$$E(u(t)) := \mathrm{Im} \int_{\mathbf{T}} \partial_x^2 u \, \overline{\partial_x u} \, dx + \frac{\beta}{2}\|\partial_x u\|_{L^2}^2 + \frac{1}{3}\|u\|_{L^4}^4 = E(u_0).$$

Remark 1 The energy E is NOT positive definite even if the size of the mass is small. This is a sharp contrast to the cubic NLS. We can not expect the Gibbs measure for (1).

- Gaussian Measures

Gaussian measure μ_s, $s \in \mathbf{R}$ with covariance operator $2(I-\Delta)^{-s}$ is formally defined as

$$d\mu_s = \prod_{k \in \mathbf{Z}} Z_{s,k}^{-1} e^{-\frac{1}{2}\langle k \rangle^{2s}(a_k^2+b_k^2)}\ da_k db_k, \quad (3)$$

$$\langle k \rangle := (1+|k|^2)^{1/2}.$$

da_k, db_k ; Lebesgue measures on $\mathbf{R}$,
$Z_{s,k}$; normalization constants.

Let $\hat{u}(k)$ denote the k-th Fourier coefficient of u and

$$X_N = \Big\{ v = \sum_{|k| \leq N} (a_k + i b_k) \frac{e^{ikx}}{\sqrt{2\pi}} \;\Big|\; a_k, b_k \in \mathbf{R} \Big\},$$

$$P_N : L^2(\mathbf{T}) \to X_N, \text{ orthogonal projection.}$$

Remark 2 Restriction $\mu_{s,N}$ of μ_s to X_N is the $2N+1$ dimensional Gaussian distribution and μ_s can be defined as

$$\mu_s := \lim_{N\to\infty} \mu_{s,N},$$

which is a measure on the space

$$X = \bigcap_{l<s-1/2} H^s(\mathbf{T}) \subset H^{s-1/2}(\mathbf{T}).$$

Another way to define Gaussian measures is as follows.
$(\Omega, \mathcal{M}, P)$; probability space,

$\{g_n(\omega)\}_{n=-\infty}^{\infty}$; sequence of independent standard complex Gaussian variables on $(\Omega, \mathcal{M}, P)$.
Define random variable u as

$$\omega \mapsto u(x;\omega) = \sum_{k\in\mathbf{Z}} \frac{g_k(\omega)}{\langle k\rangle^s} \frac{e^{ikx}}{\sqrt{2\pi}}. \qquad (4)$$

Gaussian measure μ_s is the distribution that u obeys.

Remark 3 How does the infinite product of

(3) or the series of (4) make sense? In other words, in which function space does the Gaussian measure live? This is equivalent to the following question: For what σ does u given by (4) belong to $H^\sigma(\mathbf{T})$? In fact,

$$\sigma < s - \frac{1}{2} \implies u \in H^\sigma(\mathbf{T}) \; a.s.$$

$$\mathbb{E}\|u\|_{H^\sigma}^2 = \int_\Omega \sum_{k\in\mathbf{Z}} \langle k\rangle^{2(\sigma-s)} |\hat{u}(k;\omega)|^2 \, dP_\omega$$

$$= \sum_{k \in \mathbf{Z}} \langle k \rangle^{2(\sigma - s)} \mathbb{E}|g_k|^2$$

$$= \sum_{k \in \mathbf{Z}} \langle k \rangle^{2(\sigma - s)} < \infty \text{ if } \sigma - s < -1/2.$$

Furthremore, for the d-dimensional case, the support of μ_s is $H^\sigma(\mathbf{T}^d)$ with $\sigma < s - d/2$.

Example 1 When $s = 1$, μ_1 is called the "Wiener" measure.

- Invariant Gibbs Measure

$$i\partial_t u = -\partial_x^2 u + |u|^2 u, \tag{5}$$

$$t \in \mathbf{R}, \quad x \in \mathbf{T}^2,$$

$$u(0,x) = u_0(x), \quad x \in \mathbf{T}^2. \tag{6}$$

$$d\mu_G := Z^{-1} e^{-\frac{1}{4}\|u\|_{L^4}^4} \, d\mu_1$$

$$= \lim_{N\to\infty} \Big\{ Z_N^{-1} e^{-\frac{1}{4}\|P_N u\|_{L^4}^4}$$

$$\times \prod_{|k|\leq N} e^{-\frac{1}{2}\langle k\rangle^2 (a_k^2+b_k^2)}\ da_k db_k \Big\},$$

$$a_k = \operatorname{Re}\hat{u}(k),\ b_k = \operatorname{Im}\hat{u}(k) \in \mathbf{R}^2,$$

$$u \in \operatorname{supp}\mu_G = \cap_{s<0} H^s(\mathbf{T}) \subset H^\sigma(\mathbf{T}), \sigma < 0.$$

[B1] Bourgain, 1996, cubic NLS on $\mathbf{T}^2$
LWP in $X \supset \operatorname{supp}\mu_G \Longrightarrow$ a.s. GWP in X
and $\exists$ invariant Gibbs measure,
where X is a function space $\supset L^2(\mathbf{T})$.

[B2] Bourgain, 1997, KdV and mKdV on $\mathbf{T}$, $\exists$ invariant Gibbs measure in $X \supset H^{1/2}(\mathbf{T})$.

Before Bourgain's work in 1996, it was known that one of either a.s. GWP in X or $\exists$ Gibbs measure yields another. But Bourgain proved the both assertions simultaneously by using LWP in X. Furthermore, as a result of his proof, growth rate of the X norm of solutions is $O(\sqrt{\log t})$ as $t \to \infty$.

Remark 4 Gibbs measure seems to be natural and important. But it lives on rough function spaces. This excludes an interesting class of solutions, for example, the measure of the set of all finite energy solutions is zero. Furthermore, not all Hamiltonian systems have the Gibbs measure like NLS (1) with third order dispersion.

- Quasi-Invariance of Gaussian Measures

Quasi-invariance might be able to replace invariance. If it would be the case, quasi-invariant measures would be useful to study a class of solutions which invariant Gibbs measures neglect.

Aim: Prove the quasi-invariance of the Gaussian measure with the explicit formula of the Radon-Nikodym derivative. Namely, prove the Gaussian measure is mutually absolutely continuous with the Gaussian measure transported by (1). If it is the case, what is the Radon-Nikodym derivative like?

Definition 1 $(X, \mathcal{M}, \mu)$; measure space, Mapping $T : X \to X$ is said to be measurable if $T^{-1}A \in \mathcal{M}$ for $A \in \mathcal{M}$. For a measure μ on X, the pushforward measure $T_*\mu$ is defined as $\mu(T^{-1}A)$ for $A \in \mathcal{M}$.
(i) μ is said to be invariant under T if $T_*\mu(A) = \mu(A)$ for $A \in \mathcal{M}$.
(ii) μ is said to be quasi-invariant under T if

$T_*\mu$ and μ are equivalent, i.e., mutually absolutely continuous with respect to each other.

Related Results to Our Problem

Quasi-Invariance, Gaussian Measure and Formula of Radon-Nikodym Derivative are referred to as QI, GM and FRND, respectively.

[Ku] H.-H. Kuo (1971), QI of GM with FRND for general nonlinear maps which have the form $I + H$, where H is of trace class.

[R] Ramer (1974), QI of GM with FRND for general nonlinear maps which have the form $I + H$, where H and DH are Hilbert-Schdmidt.

[Cr] Cruzeiro (1983), QI of GM with FRND under flows of ODEs driven by vector fields.

[Tz] Tzvetkov (2015), QI of GM for 1D nonlinear Hamiltonian PDEs.

(4th order NLS) Oh and Tzvetkov (2017), Oh, Sosoe and Tzvetkov (2018),

(Semilinear wave equations) Sosoe, Trenberth and X. Xiao (2019),
(3rd order NLS) Oh, Tsutsumi and Tzvetkov (2019), QI of GM for $\alpha > 3/4$

[BT] Burq and Thomann (2020),
arXiv:2012.13571v1,
almost sure scattering for 1D NLS by using quasi-invariance of Gaussian measure.

- Main Results

Theorem 1 (Debussche-Y.T, 2021) (NR), $\alpha > 1/2$, $R > 0 \Longrightarrow$ Gaussian measure $\chi_{\{\|u_0\|_{L^2(\mathbf{T})} \le R\}}\mu_\alpha(du_0)$ with L^2 cut-off is quasi-invariant under the flow generated by the third-order cubic NLS (1).

We next consider the finite dimensional approximation to (1). Let u_N be a solution of the following Cauchy problem:

$$\partial_t u_N = -i(i\partial_x^3 + \beta\partial_x^2)u_N \tag{7}$$

$$-iP_N\big(|u_N|^2u_N\big),\ t\in\mathbf{R},\ x\in\mathbf{T},$$

$$u_N(0,x) = P_Nu_0(x), \quad x\in\mathbf{T}. \tag{8}$$

Put $(\cdot,\cdot) = \mathrm{Re}\,(\cdot,\cdot)_{L^2(\mathbf{T})}$ and regard $L^2(\mathbf{T})$ as a real Hilbert space. Let $D = (-\partial_x^2)^{1/2}$. Set

$$f_N(t,u_0) = \chi_{\{\|u_0\|_{L^2(\mathbf{T})}\le R\}} \tag{9}$$

$$\times\exp\Big(-\int_0^t \big(i(|u_N|^2u_N)(-r,u_0),$$

$$D^{2\alpha}u_N(-r,u_0)\big)\ dr\Big),$$

$$f(t,u_0) = \chi_{\{\|u_0\|_{L^2(\mathbf{T})} \leq R\}} \times \exp\Big(- \int_0^t (i(|u|^2 u)(-r,u_0), D^{2\alpha} u(-r,u_0))\ dr \Big). \tag{10}$$

Theorem 2 (Debussche-Y.T, 2021) $R > 0$, $\alpha > 1/2$, $0 < s < \alpha - 1/2$, $u_0 \in H^s(\mathbf{T})$. $\{f_N(t,u_0)\}$ is uniformly bounded in $N \in \mathbf{N}$ in $L^p(d\mu_\alpha)$ for some $p > 1$. $\implies$ $f(t,u_0)$ is in $L^p(d\mu_\alpha)$ and is the Radon-Nikodym derivative at time t of the L^2 cut-off Gaussian measure transported by (1).

- Sketch of Proofs for Thoerems 1 and 2

  We now consider the Liouville equation with respect to the Radon-Nikodym derivative associated with the Gaussian measure transported by the finite dimensional approximation (7).

  $\nu_{N,\alpha,R} := \chi_{\{\|u_N\|_2 \le R\}}\mu_\alpha$,

  φ; C^1 cylinder function with compact support in the open ball of radius R,

$g_N(t, u_{0,N})$; Radon-Nikodym derivative of $\nu_{N,\alpha,R}$ transported by (7),

$$\frac{d}{dt}\int \varphi(u_{0,N})g_N(t,u_{0,N})d\nu_{N,\alpha,R}(u_{0,N})\Big|_{t=t_0}$$
$$= \int \varphi(u_{0,N})\Big(\big[i\big(i\partial_x^3+\beta\partial_x^2\big)u_{0,N} \qquad (11)$$
$$+ P_N\big(i|u_{0,N}|^2u_{0,N}\big)\big],$$
$$\nabla_{u_{0,N}}g_N(t_0,u_{0,N})\Big)\ d\nu_{N,\alpha,R}(u_{0,N})$$

$$- \int \varphi(u_{0,N}) \left(D^{2\alpha} u_{0,N}, P_N\left(i|u_{0,N}|^2 u_{0,N}\right)\right)$$
$$\times g_N(t_0, u_{0,N}) \; d\nu_{N,\alpha,R}(u_{0,N}), \quad t_0 > 0 \tag{12}$$

where $\nabla_{u_{0,N}}$ is gradient with respect to $u_{0,N}$. Here, we have used the following cancellation:

$$\begin{aligned} \mathrm{Tr}\Big[\mathrm{div}_{u_{0,N}} \Big(i(i\partial_x^3 + \beta\partial_x^2) u_{0,N} \\ + iP_N\big(|u_{0,N}|^2 u_{0,N}\big)\Big)\Big] \\ = \mathrm{Tr}\Big[\mathrm{div}_{u_{0,N}} J \nabla_{u_{0,N}} H(u_{0,N})\Big] = 0. \end{aligned}$$

[dBBF] de Bouard, Debussche and Fukuizumi, SIAM J. Math. Anal., **56** (2018), on the last line of page 5900.

Since φ is an arbitrary C^1 cylinder function, (12) yields

$$\frac{d}{dt} g_N(t+t_0, u_N(t, u_{0,N}))\Big|_{t=0} = \Big(\big\{\partial_x^3 u_{0,N} + P_N\big(|u_{0,N}|^2 u_{0,N}\big)\big\}, \nabla_{u_{0,N}} g_N(t_0, u_{0,N})\Big) - \Big(D^{2\alpha} u_{0,N}, P_N\big(|u_{0,N}|^2 u_{0,N}\big)\Big) g_N(t_0, u_{0,N}).$$

Let $t_0 > 0$ be fixed. We write $g_N(t, u_{0,N})$ $(0 \leq t \leq t_0)$ for $g_N(t+t_0, u_N(t, u_{0,N}))$ $(-t_0 \leq t \leq 0)$ by changing the variables $t + t_0 \mapsto t$. The above equation of $g_N(t_0, u_{0,N})$ with $g_N(0, \cdot) = 1$ implies the explicit formula of the density: For $t \in [0, t_0]$,

$$
\begin{aligned}
g_N(t, u_{0,N}) =& \exp\Big\{-\int_0^t \big(D^{2\alpha} u_N(-r, u_{0,N}), \\
& \qquad i\big(|u_N|^2 u_N\big)(-r, u_{0,N})\big) dr\Big\} \\
=& f_N(t, u_{0,N}). \qquad (13)
\end{aligned}
$$

This leads to Theorem 2.

Remark 5 The formula (13) seems to be natural, because we have the L^2 energy estimate as follows.

$$
\begin{aligned}
&\frac{1}{2}\|D^\alpha u(t)\|_2^2 - \frac{1}{2}\|D^\alpha u_0\|_2^2 \\
&= -\int_0^t \left(D^\alpha(i|u|^2 u)(r), D^\alpha u(r)\right)\, dr.
\end{aligned}
$$

We note that the first and the second terms on the left side correspond to the weights of the Gaussian measures at time t and at 0 with respect to the Lebesgue measure, respectively (see (3) on page 4).

All what we have to do for the proof of Theorem 1 is to bound the time integral in (13) by the $H^{\alpha-1/2-\varepsilon}$ norm of the solution u, where ε is a small positive constan. This can be done by the dispersive PDE smoothing effect for equation (1). In fact,

$$\text{time average} \implies \text{spatial regularity.}$$

Thus, Theorem 2 and the following lemma yield Theorem 1.

For $r \geq 1$ and $B > 0$, we set

$$F(\omega) := \chi_{\left\{ \left(\sum_{n\in\mathbf{Z}} |g_n(\omega)|^2/\langle n\rangle^{2\alpha} \right)^{1/2} < B \right\}} \times \exp\left(\Big\| \sum_{n\in\mathbf{Z}} \frac{g_n(\omega)}{\langle n\rangle^{\alpha}} e^{inx} \Big\|^r_{B^s_p(\mathbf{T})} \right),$$

where $\{g_n\}$ is a sequence of independent equidistributed complex centered Gaussian random variables.

Lemma 1 (Debussche-Y.T, 2021) Let $\alpha > 0$, $p \geq 2$ and $s \geq 0$ be such that $\alpha - 1 + 1/p > s$. Assume that $B > 0$ for $r < \frac{4\alpha p}{p-2+2ps}$ and B is sufficiently small for $r = \frac{4\alpha p}{p-2+2ps}$. Then, $F(\omega) \in L^1(d\omega)$.

The proof of Lemma 1 follows from Bourgain's argument (1996).

Debussche and Tsutsumi, J. Funct. Anal., **281** no.3 (2021).

Thank you for your attention!

ご清聴ありがとうございました.