On the heat equation with a dynamic singular potential

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in collaboration with

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PDEs and Probability Theory - beyond boundaries -
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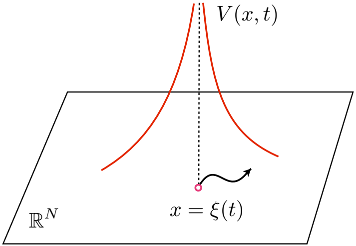
[Heat equation with a singular potential]

$$u_t = \Delta u + V(x,t)u, \qquad x \in \mathbb{R}^N \setminus \{\xi(t)\}.$$

where $u = u(x,t)$, $N > 2$.

The potentisl V is assumed to be singular at $\xi(t)$: $V(x,t) \to \infty$ as $x \to \xi(t)$.



[Heat equation with a singular potential]

$$u_t = \Delta u + V(x,t)u, \qquad x \in \mathbb{R}^N \setminus \{\xi(t)\}.$$

where $N > 2$.

Topics:

- Existence and non-existence of positive solutions
- Optimal class of initial values for solvability

- Lower and upper estimates
- Asymptotic profile around singularities
- Classification of solutions

- Critical values of parameters

Plan of my talk:

I: Fixed singularity $V(x,t) = \dfrac{\lambda}{|x|^2}$

PDE and probabilistic approach by Baras-Goldstein (1984)

Critical value $\lambda = \lambda_c(N)$ for existence

II: Moving singularity $V(x,t) \lesssim \dfrac{\lambda}{|x-\xi(t)|^2}$

PDE approach by Chern-Hwang-Takahashi-Y (2021)

Extension of Baras-Goldstein

III: Asymptotics of solutions $V(x,t) \sim \dfrac{\lambda}{|x-\xi(t)|^2}$

PDE approach by Takahashi-Y

Classification of singularities of solutions

IV: Fractional Brownian motion of $\xi(t)$ $V(x,t) = \dfrac{\lambda}{|x-\xi(t)|^{\mu}}$

Probabilistic approach by Okada-Y

Critical value $\mu = \mu_c(H)$ (H: the Hurst exponent)

[Part I: Fixed singularity]

... PDE and probabilistic approach by Baras-Goldstein (1984)

Critical value $\lambda = \lambda_c(N)$ for existence

Elliptic equation with the Hardy potential:

$$\Delta u + \frac{\lambda}{|x|^2} u = 0, \qquad x \in \mathbb{R}^N \setminus \{0\}.$$

(Many studies have been done.)

<u>Radial solutions</u>

Assume $N > 2$. Substituting $u = r^{-\alpha}$, $r := |x|$, we have

$$u_{rr} + \frac{N-1}{r} u_r + \frac{\lambda}{r^2} u = \left\{ \alpha^2 - (N-2)\alpha + \lambda \right\} r^{-\alpha-2}.$$

Hence $u = r^{-\alpha}$ is a solution if

$$\alpha^2 - (N-2)\alpha + \lambda = 0.$$

- Subcritical case: If $\lambda < \lambda_c = \dfrac{(N-2)^2}{4}$, the quadratic equation

$$\alpha^2 - (N-2)\alpha + \lambda = 0.$$

has two real roots:

$$0 < \alpha_1 < \frac{N-2}{2} < \alpha_2 < N-2,$$

and there are two types of positive radial singular solutions:

$$u = C|x|^{-\alpha_1} \qquad \text{(weak singularity)}$$

$$u = C|x|^{-\alpha_2} \qquad \text{(strong singularity)}$$

- Supercritical case: If

$$V(x) > \frac{\lambda_c}{|x|^2}$$

then there are no positive radial solutions for

$$\Delta u + V(x)u = 0, \qquad x \in \mathbb{R}^N \setminus \{0\}.$$

Heat equation with the Hardy potential

$$u_t = \Delta u + \frac{\lambda}{|x|^2} u, \qquad x \in \mathbb{R}^N \setminus \{0\}.$$

Baras-Goldstein (1984) showed that $\lambda_c := \frac{(N-2)^2}{4} > 0$ is critical.

Theorem (Critical value for existence)

(i) If $0 < \lambda < \lambda_c$, there exists a positive global solution.

(ii) If $\lambda > \lambda_c$, there exists no positive solution.

... by Energy method, Feynman-Kac formula

Many other works on

- Existence of solutions.
- Optimal class of initial values.

Question

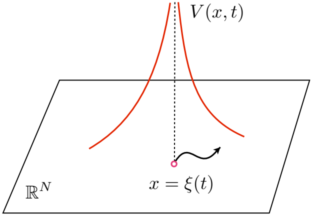
What if the singular point $\xi(t)$ moves in time?

Heat equation with a dynamic singular potential

$$u_t = \Delta u + V(x,t)u, \qquad x \in \mathbb{R}^N \setminus \{\xi(t)\},$$

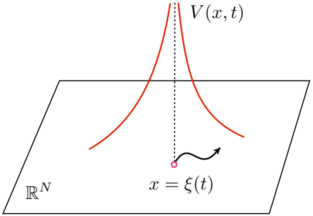
where V has a singularity at $\xi(t)$.

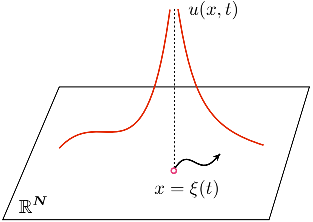
$$\text{ex.} \qquad V(x,t) \simeq \frac{\lambda}{|x - \xi(t)|^{\mu}}.$$



Interaction between $V(x,t)$ and $u(x,t)$ is more delicate if $\xi(t)$ moves.

$$u_t = \Delta u + V(x,t)u(x,t)$$





[Part II: Moving singularity]

... PDE approach by Chern-Hwang-Takahashi-Y (2021)

Extension of Baras-Goldstein

Initial value problem

$$\text{(IVP)} \quad \begin{cases} u_t = \Delta u + V(x,t)u, & x \in \mathbb{R}^N \setminus \{\xi(t)\},\ t \in (0,T], \\ u(x,0) = u_0(x), & x \in \mathbb{R}^N \setminus \{\xi(0)\}. \end{cases}$$

Basic assumptions:

(A1) $V(x,t)$ is nonnegative and continuous in $(x,t) \in \mathbb{R}^N \setminus \{\xi(t)\} \times [0,\infty)$.

$V(x,t)$ is singular at $\xi(t)$ (i.e., $V(x,t) \to \infty$ as $x \to \xi(t)$).

$V(x,t)$ is bounded for $|x - \xi(t)| > 1$.

(A2) $\xi(t)$ is γ-Hölder continuous in $t \geq 0$ with $\gamma > 1/2$.

(A3) $u_0(x) \in C(\mathbb{R}^N \setminus \{\xi(0)\})$, $u_0(x) \geq 0, \not\equiv 0$ for $x \neq \xi(0)$.

$u_0(x)$ is bounded for $|x - \xi(0)| > 1$.

Def. Minimal solution

Define

$$V_n(x,t) := \min\{V(x,t), n\}.$$

If $u_0 \in L^1(\mathbb{R}^N)$, then for each $n \in \mathbb{N}$, there exists a unique bounded solution of the following regular problem:

$$\begin{cases} u_t(x,t) = \Delta u(x,t) + V_n(x,t)u, & x \in \mathbb{R}^N, \quad t > 0, \\ u(x,0) = u_0(x), & x \in \mathbb{R}^N. \end{cases}$$

We denote the unique solution by $u_n(x,t)$. If

$$u(x,t) := \lim_{n\to\infty} u_n(x,t), \qquad x \neq \xi(t),$$

exists, then the limiting function $u(x,t)$ satisfies (IVP). We call such $u(x,t)$ a minimal solution (or proper solution). For the existence of a minimal solution, it suffices to find an upper bound.

$$(\text{IVP}) \quad \begin{cases} u_t = \Delta u + V(x,t)u, & x \neq \xi(t),\ t \in (0,T] \\ u(x,0) = u_0(x), & x \neq \xi(0). \end{cases}$$

Theorem (Existence of a solution)

Assume that V satisfies

$$0 \leq V(x,t) \leq \frac{\lambda}{|x-\xi(t)|^2}, \qquad |x-\xi(t)| < R,\ t \in [0,T]$$

with some $\lambda \in (0, \lambda_c)$ and $R > 0$. If

$$0 \leq u_0(x) \leq C_1 |x-\xi(0)|^{-k}, \qquad |x-\xi(0)| < R$$

with some $k < \alpha_2 + 2$ and $C_1 > 0$, then (IVP) has a minimal solution satisfying

$$u(x,t) \leq C_2 |x-\xi(t)|^{-\alpha_1-\varepsilon}, \qquad |x-\xi(t)| < R,\ t \in [\tau, T],$$

where $\varepsilon > 0, \tau > 0$ are arbitrary, $C_2 = C_2(\varepsilon,\tau) > 0$ is a constant.

Theorem 2 (Lower bound)

Assume that V satisfies

$$V(x,t) > \frac{\lambda}{|x-\xi(t)|^2}, \qquad |x-\xi(t)| < R,\ t \in [0,T]$$

with some $\lambda \in (0, \lambda_c)$ and $R > 0$. Then any solution of (IVP) satisfies

$$u(x,t) \geq C|x-\xi(t)|^{-\alpha_1+\varepsilon}, \qquad |x-\xi(t)| < R,\ t \in [\tau,T],$$

where $\varepsilon, \tau > 0$ are arbitrary, $C = C(\varepsilon,\tau) > 0$ is a constant.

Key observation for the proof

Rewrite the equation into an integral equation

$$u(x,t) = \int_{\mathbb{R}^N} G(x,y,t)u_0(y)dy + \int_0^t \int_{\mathbb{R}^N} G(x,y,t-s)V(y,s)u(y,s)dyds,$$

where G is the heat kernel given by

$$G(x,y,t) = \frac{1}{(4\pi t)^{N/2}} \exp\Big(- \frac{|x-y|^2}{4t} \Big).$$

Set

$$\tilde{x} = x - \xi(t), \quad \tilde{y} = y - \xi(s),$$

$$\tilde{u}(\tilde{x},t) = u(x+\xi(t)), \qquad \tilde{V}(\tilde{x},t) = V(\tilde{x}+\xi(t),t)$$

to transform it to the case of a fixed singularity.

Then we have

$$\tilde{u}(\tilde{x},t) = \int_{\mathbb{R}^N} G(x,y,t)u_0(y)dy$$
$$+ \int_0^t \int_{\mathbb{R}^N} G(\tilde{x}+\xi(t),\tilde{y}+\xi(s),s)\tilde{V}(\tilde{y},s)\tilde{u}(\tilde{y},s)d\tilde{y}ds.$$

By the γ-Hölder continuity of $\xi(t)$ with $\gamma > 1/2$, the heat kernel satisfies

$$(1+\delta)G\left(\tilde{x},\tilde{y},\frac{t-s}{1+\delta}\right) \geq G(\tilde{x}+\xi(t),\tilde{y}+\xi(s),s) \geq (1-\delta)G\left(\tilde{x},\tilde{y},\frac{t-s}{1-\delta}\right)$$

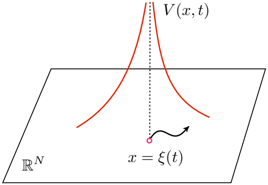
with some small $\delta > 0$.

Using these inequalities, we obtain integral inequalities for u_n. Then Gronwall's inequalities yield upper and lower estimate of u_n.

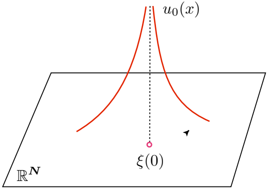
Non-existence for large initial data

The conditions $\lambda < \lambda_c$ and $k < \alpha_2 + 2$ are essential for the existence.

$$u_t = \Delta u + V(x,t)u(x,t)$$



$$V(x,t) \simeq \frac{\lambda}{|x - \xi(t)|^2}$$



$$u_0(x) \simeq C|x - \xi(0)|^{-k}$$

Theorem (Non-existence for large initial data)

Assume that V satisfies

$$V(x,t) \geq \frac{\lambda}{|x-\xi(t)|^2}, \qquad |x-\xi(t)| < R,\ \ t \in [0,T]$$

with some $\lambda \in (0, \lambda_c)$ and $R > 0$. If

$$u_0(x) \geq C|x-\xi(0)|^{-k}, \qquad |x-\xi(0)| < R$$

with some $k > \alpha_2 + 2$ and $C > 0$, then (IVP) has no solution.

Theorem (Nonexistence in the supercritical case)

Assume that V satisfies

$$V(x,t) \geq \frac{\lambda}{|x-\xi(t)|^2}, \qquad |x-\xi(t)| < R,\ \ t \in [0,\tau]$$

with some $\lambda > \lambda_c$, $R > 0$, $\tau \in (0,T)$. Then (IVP) has no solution for any initial data.

Proof of the non-existence

By using Theorem (Lower bound), we can show that the integral operator

$$I[u] := \int_{\mathbb{R}^N} G(x, y, t)u_0(y)dy + \int_0^t \int_{\mathbb{R}^N} G(x, y, t-s)V(y, s)u(y, s)dyds$$

satisfies $I[u] > u$. Hence there is no fixed point (no solution).

[Part III: Asymptotics of solutions]

... PDE approach by Takahashi-Y

Classification of singularities of solutions

$$
\text{(IVP)} \quad \begin{cases} u_t = \Delta u + V(x,t)u, & x \in \mathbb{R}^N \setminus \{\xi(t)\},\ t \in (0,T], \\ u(x,0) = u_0(x), & x \in \mathbb{R}^N \setminus \{\xi(0)\}. \end{cases}
$$

Assume that V is expanded as

$$
V(x,t) = \frac{\lambda}{|x-\xi(t)|^2} + o(|x-\xi(t)|^{-2+\delta})
$$

as $x \to \xi(t)$ uniformly in $t \in [0,T]$, where $\delta > 0$ and $\lambda > 0$.

Problem

Study the asymptotic behavior of solutions as $x \to \xi(t)$.

We already proved in Theorem (Existence of a solution) and Theorem (Lower bound) that if $0 < \lambda < \lambda_c$ for $t \in [0, T]$, then the minimal solution of (IVP) satisfies

$$C_1|x - \xi(t)|^{-\alpha_1+\varepsilon} \leq u(x,t) \leq C_2|x - \xi(t)|^{-\alpha_1-\varepsilon},$$

where $\varepsilon > 0$ and $C_1, C_2 > 0$ are constants.

The minimal solution is unique, but there exist larger solutions.

Existence of a larger solution

Theorem (Larger solution)

Assume $\lambda \in C^1([0,T])$ and $0 < \lambda < \lambda_c$. Let $h \in C^1([0,T])$ be an arbitrary positive function. If

$$u_0(x) = h(0)|x - \xi(0)|^{-\alpha_2} + O(|x - \xi(0)|^{-\alpha_2 + \mu}) \quad \text{as } x \to \xi(0)$$

for some $\mu > 0$, then (IVP) has a solution satisfying

$$u(x,t) = h(t)|x - \xi(t)|^{-\alpha_2} + O(|x - \xi(t)|^{-\alpha_2 + \mu'}) \quad \text{as } x \to \xi(t)$$

for every $t \in [0,T]$, where $0 < \mu' < \mu$.

- The larger solution is asymptotically radially symmetric as $x \to \xi(t)$ for every t.
- Switching to a minimal solution is possible at any time.

Idea of the proof

We first consider the heat equation

$$U_t = \Delta U + \delta(x - \xi(t)),$$

where δ is a Dirac measure. The equation has a solution expressed as

$$U = C \int_{-1}^{t} G(x, \xi(s), t - s) ds, \qquad C = N(N-2)|B|.$$

which satisfies

$$U(x,t) \sim C|x - \xi(t)|^{-(N-2)}.$$

... Takahashi-Y (2015)

We also found that $u \simeq U^{\alpha_i/(N-2)}$ is a very nice approximate solution. We use this to construct suitable comparison functions.

Using the solution U of the heat equation, we construct a supersolution of the form

$$u^+(x,t) := k(t)U(x,t)^{\alpha_2/(N-2)} + U(x,t)^{(\alpha_2-\mu')/(N-2)} + R(x,t),$$

and a subsolution of the form

$$u^-(x,t) := k(t)U(x,t)^{\alpha_2/(N-2)} - U(x,t)^{(\alpha_2-\mu')/(N-2)} - R(x,t),$$

where $R(x,t)$ is a suitable bounded function. Namely,

$$u_t^+ > \Delta u^+ + \frac{\lambda}{|x-\xi(t)|^2} u^+,$$

$$u_t^- < \Delta u^+ + \frac{\lambda}{|x-\xi(t)|^2} u^-.$$

Then the comparison principle implies that there exists a solution between u^+ and u^-.

Critical case $\lambda = \lambda_c$

Theorem (Critical case)

Assume $\lambda = \lambda_c$. If

$$0 < u_0(x) \le K \left\{ 1 + |x - \xi(0)|^{-\alpha_1} \left(\log \left(e + \frac{1}{|x - \xi(0)|} \right) \right)^{\beta} \right\},$$

for some $K > 1$, $0 < \beta < 1$, then (IVP) has a solution satisfying

$$C_1 |x - \xi(t)|^{-\alpha_1} \le u(x,t) \le C_2 |x - \xi(t)|^{-\alpha_1} \left(\log \left(e + \frac{1}{|x - \xi(t)|} \right) \right)^{\beta}$$

with some constants $C_1, C_2 > 0$.

We take a supersolution and a subsolution of the form

$$u^{\pm}(x,t) := CU(x,t)^{(\alpha_1)/(N-2)} (\log(e + U(x,t)))^{-\beta} \pm b(t),$$

where $b(t)$ is a suitable bounded function

Theorem (Classification)

Assume $0 < \lambda < \lambda_c$ for $t \in [0, T]$.

(i) Suppose that a solution u satisfies

$$u(x,t) \leq K|x - \xi(t)|^{-\alpha_2+\varepsilon}, \qquad |x - \xi(0)| < R,$$

where $\varepsilon > 0$ and $R > 0$ are arbitrary and $K > 0$. Then u is a minimal solution:

$$C_1|x - \xi(t)|^{-\alpha_1+\varepsilon} \leq u(x,t) \leq C_2|x - \xi(t)|^{-\alpha_1-\varepsilon}$$

(ii) Suppose that a solution u satisfies

$$u(x,t) \geq K|x - \xi(t)|^{-\alpha_1-\varepsilon}, \quad 0 < |x - \xi(t)| < R,$$

where $\varepsilon > 0$, $K > 0$ and $R > 0$. Then u satisfies

$$C_1|x - \xi(t)|^{-\alpha_2+\varepsilon} \leq u(x,t) \leq C_2|x - \xi(t)|^{-\alpha_2-\varepsilon}.$$

Idea of the proof

By a careful estimate of the integral

$$I[u] = \int_{\mathbb{R}^N} G(x, y, t)u_0(y)dy + \int_0^t \int_{\mathbb{R}^N} G(x, y, t - s)V(y, s)u(y, s)dyds,$$

we can show that $I[u]$ has a fixed point only in the case as assumed.

In fact,

$$0 < u(x, t) \le C|x - \xi(t)|^{-\alpha_1+\varepsilon} \quad \Longrightarrow I[u] > u$$

$$C_1|x - \xi(t)|^{-\alpha_1-\varepsilon} \le u(x, t) \le C_2|x - \xi(t)|^{-\alpha_2+\varepsilon} \quad \Longrightarrow I[u] < u$$

$$C_1|x - \xi(t)|^{-\alpha_2-\varepsilon} \le u(x, t) \le C_1|x - \xi(t)|^{-\alpha_2-2+\varepsilon} \quad \Longrightarrow I[u] > u$$

[Part IV: Fractional Brownian motion of $\xi(t)$]

Probabilistic approach by Okada-Y

Critical value $\mu = \mu_c(H)$ (H: the Hurst exponent)

$$u_t = \frac{1}{2}\Delta u + V(x,t)u, \qquad x \in \mathbb{R}^N \setminus \{\xi(t)\}, \quad 0 < t < T,$$

where

$$V(x,t) = \frac{\lambda}{|x - \xi(t)|^{\mu}}.$$

Assume that $\xi(t)$ is a sample path of the Fractional Brownian motion. Then $u(x,t)$ can be regarded as a random variable.

Fractional Brownian motion

Fractional Brownian motion $\{B^H(t)\}_{t\geq 0}$ with the Hurst exponent $0 < H < 1$ is the Gaussian process specified by

(i) $B(0)^H = 0$.

(ii) $E[B^H(t)] = 0$ for $t \geq 0$.

(iii) $E[B^H(t)B^H(s)] = \frac{1}{2}\left(|t|^{2H} + |s|^{2H} - |t-s|^{2H}\right)$ for $t, s \geq 0$.

- Self-similar process

$$E[B^H(\beta t)^2] = |\beta|^{2H} E[B^H(t)^2], \quad E[B^H(\beta t)B^H(\beta s)] = |\beta|^{2H} E[B^H(t)B^H(s)]$$

- Sample path is $(H-\varepsilon)$-Hölder continuous in $t > 0$ a.s, a.e.

$\Longrightarrow$ If $H > 1/2$, then there exists a positive solution.

- $H = 1/2$ corresponds to the standard Brownian motion.

Theorem (The case $H < 1/2$)

Assume that $\xi(t)$ is a sample path of the fractional Brownian motion with the Hurst exponent $0 < H < 1/2$.

(i) If $\mu \geq (1/H) \wedge N$, then the equation has no positive solution.

(ii) If $2 < \mu < (1/H) \wedge N$, then the equation has a positive solution.

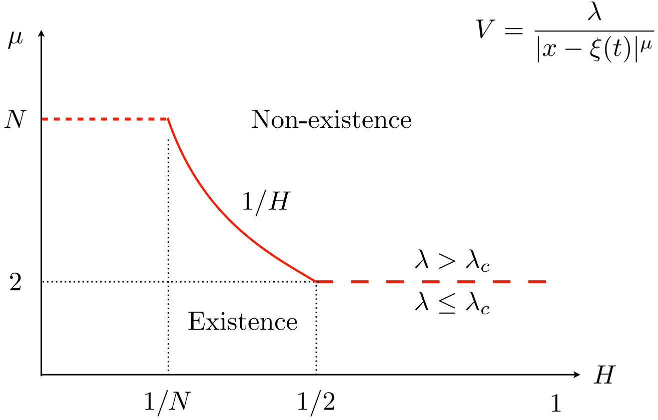
(i) If $\mu \geq (1/H) \wedge N$, then

$$P_\xi(u(x,t) = \infty \text{ for all } x \in \mathbb{R}^N \setminus \{\xi(t)\}) = 1$$

for every $t > 0$.

(ii) If $0 < \mu < (1/H) \wedge N$, then

$$P_\xi(\forall r > 0,\ \exists C > 0 \text{ s.t. } u(x,t) \leq C \text{ for all } (t,x) \notin \mathcal{N}_r(0,0)) = 1.$$


$V = \frac{\lambda}{|x-\xi(t)|^{\mu}}$
μ
N
Non-existence
$1/H$
$\lambda > \lambda_c$
2
$\lambda \le \lambda_c$
Existence
H
$1/N$
$1/2$
1

Feynman-Kac formula

By the Feynman-Kac formula, the solution can be expressed as

$$u(x,t) = E^x\Big[u_0(B(t))\exp\big(\lambda\int_0^t |B(s)-\xi(t-s)|^{-\mu}ds\big)\Big],$$

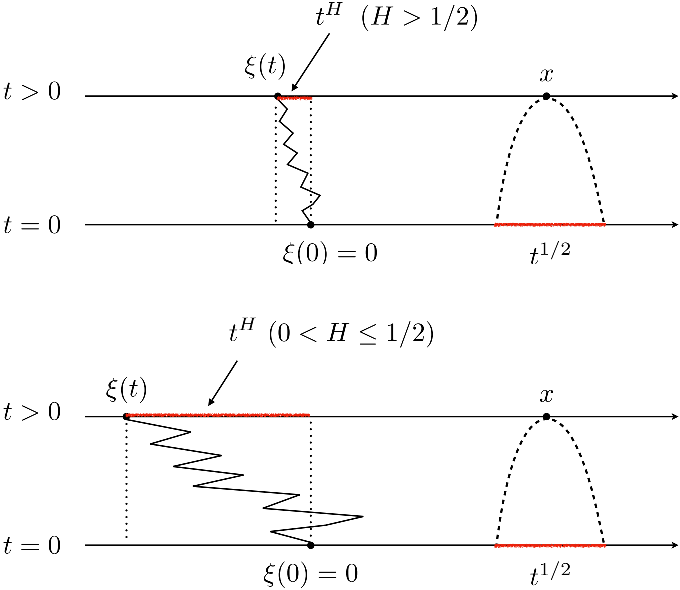
where $B(t)$ stands for the N-dimensional standard Brownian motion.

- Explicit expression
- Probabilistic techniques are available

    Properties of the fractional Brownian motion

    Chebychev's inequality, Borel-Cantelli's lemma

- Improper integral leads to a minimal solution


t^H $(H > 1/2)$
$\xi(t)$
x
$t > 0$
$t = 0$
$\xi(0) = 0$
$t^{1/2}$
t^H $(0 < H \leq 1/2)$
$\xi(t)$
x
$t > 0$
$t = 0$
$\xi(0) = 0$
$t^{1/2}$

- $H > 1/2$: Diffusion is faster than $\xi(t) \Longrightarrow$ There exists a solution.

- $1/N < H \le 1/2$: Diffusion is slower than $\xi(t)$

Then solution u satisfies the following ODE approximately:

$$\frac{d}{dt}u(x,t) \simeq |x-\xi(t)|^{-\mu}u(x,t),$$

so that

$$\int_0^t |B(s)-\xi(t-s)|^{-\mu}ds \simeq \int_0^t |\xi(t-s)|^{-\mu}ds$$
$$\approx \sum_{m=0}^{\infty}\int_0^t e^{\mu m}1_{\{e^{-m-1}\le|\xi(t-s)|\le e^{-m}\}}ds \quad \text{(occupation time)}$$

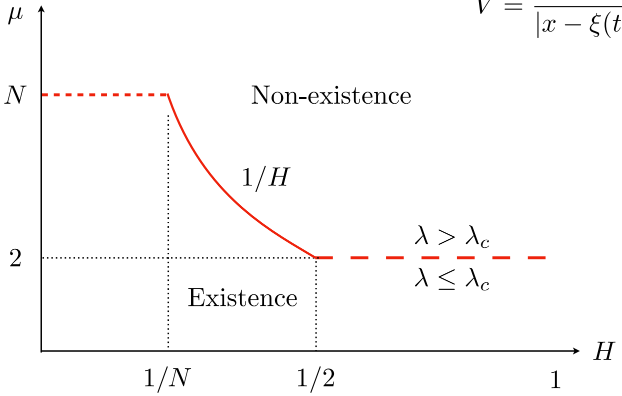
which is bounded if $\mu < 1/H$.

- $0 < \mu \le 1/N$: Singularity of the potential is too strong.

$V(x,t) = \dfrac{\lambda}{|x-\xi(t)|^{\mu}} \notin L^1_{loc}(\mathbb{R}^N) \Longrightarrow$ No solution.

Summary

$$V = \frac{\lambda}{|x - \xi(t)|^{\mu}}$$



- Baras-Goldstein (Energy method, Feynman-Kac formula)

  Existence

  Critical value $\lambda = \lambda_c(N)$

- Chern-Hwang-Takahashi-Y (Heat kernel)

  Extension of Baras-Goldstein

  $(1/2 + \varepsilon)$-Hölder continuity of $\xi(t)$

- Takahashi-Y (Comparison method)

  Asymptotics around a singularity

  Classification of singularities

- Okada-Y (Feynman-Kac formula)

  Fractional Brownian motion with the Hurst exponent $0 < H < 1/2$

  Critical value $\mu = \mu_c(H)$

Thank you for your attention !