Large N Limit of the $O(N)$ Linear Sigma Model via Stochastic Quantization

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Joint work with Hao Shen, Scott Smith and Xiangchan Zhu

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Introduction

Gaussian free field

- Recall the free field in the Euclidean quantum field theory. The usual free field on the torus $\mathbb{T}^d$ is heuristically described by the following probability measure:

$$\nu(\mathrm{d}\Phi) = C_N^{-1}\Pi_{x\in\mathbb{T}^d}\mathrm{d}\Phi(x)\exp\Big(-\int_{\mathbb{T}^d}(|\nabla\Phi|^2 + m\Phi^2)\mathrm{d}x\Big),$$

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- This corresponds to the Gaussian measure $\nu := \mathcal{N}(0, (m - \Delta)^{-1})$ rigorously defined on $\mathcal{S}'$.
- The free field describes particles which do not interact.

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- [Catellier, Chouk 18], [Mourrat, Weber 17], [Albeverio, Kusuoka 18], [Gubinelli, Hofmanova 18, 19] , [Röckner, Zhu, Z. 17/Zhu, Z. 18]...

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Stochastic quantization on $\mathbb{T}^d$, $d = 2, 3$:

$$\mathcal{L}\Phi_i = -\frac{1}{N} \sum_{j=1}^N \Phi_j^2 \Phi_i + \xi_i,$$

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Questions: Large N limit of the dynamics Φ_i and the field ν^N?

Stochastic quantization: Da Prato-Debussche trick

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- Decompose $\Phi_i = Y_i + Z_i$ as Da Prato-Debussche trick for $d=2$

$$\begin{aligned}\mathcal{L}Z_i =& \xi_i,\\ \mathcal{L}Y_i =& -\frac{1}{N}\sum_{j=1}^N (Y_j^2Y_i + Y_j^2Z_i + 2Y_jY_iZ_j \\ &+ 2Y_j \underbrace{:Z_iZ_j:}_{\text{Wick product}} + \underbrace{:Z_j^2:}_{\text{Wick product}} Y_i + \underbrace{:Z_iZ_j^2:}_{\text{Wick product}}),\end{aligned}$$

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Difficulty for $d = 3$

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- Key point: The red terms are not well defined even we do further decomposition!
- Local well-posedness: Regularity structure theory in [Hairer 14]/ Paracontrolled distribution method in [Gubinelli, Imkeller, Perkowski 15]
- This is not enough since the stopping time may depend on N

Large N limit of the dynamics

Limiting equation and convergence of the dynamics when $d = 2$

- The dynamical linear sigma model

$$\mathcal{L}\Phi_i = -\frac{1}{N}\sum_{j=1}^{N}(\Phi_j^2 - \mathbf{E}[Z_i^2])\Phi_i + \xi_i, \quad \Phi_i(0) = \phi_i$$

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Theorem [Shen, Scott, Zhu, Z. 20]

Suppose that $d = 2$ and (ψ_i, ψ_j) are independent and have the same law and for $p > 1$ $\mathbf{E}\|\phi_i - \psi_i\|^p_{C^{-\kappa}} \to 0$, as $N \to \infty$.

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- Mean field limit/ Propagation of chaos

Idea of Proof: Uniform bounds

$\Phi_i = Z_i + Y_i$, $\Psi_i = Z_i + X_i$

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Lemma 1

It holds that for $p \geq 2$

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- dissipation weaker as $N \to \infty$

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Invariant measures

Invariant measure to Limiting equation

- The limiting equation

$$(\partial_t - \Delta + m)\Psi_i = \mathcal{L}\Psi_i = -\mu\Psi_i + \xi_i,$$

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Theorem [Shen, Scott, Zhu, Z. 20]

For $d = 1, 2$, there exists $m_0 > 0$ such that: for $m \geq m_0$, the Gaussian free field $\mathcal{N}(0, (m - \Delta)^{-1})$ is the unique invariant measure to Ψ.

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- Idea: solutions converges to each other as time goes to infinity.

Convergence of invariant measure (field)

- $O(N)$ linear sigma model:

$$\nu^N = \frac{1}{C_N} \exp\Big(-2\int_{\mathbb{T}^d} \frac{1}{2}\sum_{j=1}^N |\nabla \Phi_j|^2 + \frac{m}{2}\sum_{j=1}^N \Phi_j^2 + \frac{1}{4N}\Big(\sum_{j=1}^N \Phi_j^2\Big)^2 \mathrm{d}x\Big)\mathcal{D}\Phi,$$

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For $d = 2, 3$

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Idea of proof: $\mathbb{W}_2(\nu^{N,i}, \nu) \lesssim N^{-\frac{1}{2}}$ for $d = 3$

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- Cancelation from paraprodcts [Gubinelli, Hofmanova 18]

$$\sum_{i=1}^N \Big[\langle(\Delta - m)Y_i, Y_i\rangle - \frac{1}{N}\sum_{j=1}^N \langle 2Y_j \preccurlyeq :Z_iZ_j: + Y_i \preccurlyeq :Z_j^2:, Y_i\rangle\Big]$$

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- Paracontrolled ansatz and further decomposition

$$Y_i = \varphi_i - \frac{1}{N}(m-\Delta)^{-1}\sum_{j=1}^N (2Y_j \prec :Z_i Z_j: + Y_i \prec :Z_j^2:)$$

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$$\leq\mathbf{E}\Big(\sum_{j=1}^N\|Y_j(0)\|_{L^2}^2\Big)+C\mathbf{E}\int_0^T\Big(\sum_{i=1}^N\|Y_i\|_{L^2}^2\Big)R_N\mathrm{d}s+C.$$

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$R_N=R_N-\mathbf{E}[R_N]+\mathbf{E}[R_N]$

Observables

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Theorem [Shen, Scott, Zhu, Z. 20/ Shen, Zhu, Z. 21]

Suppose that $\Phi \simeq \nu^N$. For $\kappa > 0$, m large enough, the following result holds:

- $\frac{1}{\sqrt{N}} \sum_{i=1}^N : \Phi_i^2 :$ is tight in $B_{2,2}^{-2\kappa}$ for $d = 2$ / $B_{1,1}^{-1-\kappa}$ for $d = 3$
- $\frac{1}{N} : (\sum_{i=1}^N \Phi_i^2)^2 :$ is tight in $B_{1,1}^{-3\kappa}$ for $d = 2$
- For $d = 1, 2$,

$$\lim_{N\to\infty} \frac{1}{\sqrt{N}} \sum_{i=1}^N : \Phi_i^2 : \neq \lim_{N\to\infty} \frac{1}{\sqrt{N}} \sum_{i=1}^N : Z_i^2 :$$

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- Idea: Improved moment estimate for stationary case by independence

$$\mathbf{E}\Big[\Big(\sum_{i=1}^N \|Y_i\|_{L^2}^2\Big)^q\Big] + \mathbf{E}\Big[\Big(\sum_{i=1}^N \|Y_i\|_{L^2}^2 + 1\Big)^q \Big(\sum_{i=1}^N \|\nabla Y_i\|_{L^2}^2\Big)\Big] \lesssim 1.$$

Observables

Theorem [Shen, Scott, Zhu, Z. 20/ Shen, Zhu, Z. 21]

Suppose that $\Phi \simeq \nu^N$. For $\kappa > 0$, m large enough, the following result holds:

- $\frac{1}{\sqrt{N}} \sum_{i=1}^N : \Phi_i^2 :$ is tight in $B_{2,2}^{-2\kappa}$ for $d = 2$ $/ B_{1,1}^{-1-\kappa}$ for $d = 3$
- $\frac{1}{N} : (\sum_{i=1}^N \Phi_i^2)^2 :$ is tight in $B_{1,1}^{-3\kappa}$ for $d = 2$
- For $d = 1, 2$,

$$\lim_{N\to\infty} \frac{1}{\sqrt{N}} \sum_{i=1}^N : \Phi_i^2 : \neq \lim_{N\to\infty} \frac{1}{\sqrt{N}} \sum_{i=1}^N : Z_i^2 :$$

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- Integration by parts formula/ Dyson-Schwinger from [Kupiainen 80]

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- Other models

Thank you !