Euclidean designs and relative t-designs in Q-polynomial schemes

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Notation

$\mathbb{R}^n \supset S^{n-1}$: unit sphere centered at the origin.

$x \cdot y$: canonical inner product of $\mathbb{R}^n$

$\|x\| := \sqrt{x \cdot x}$

$\mathcal{P}(\mathbb{R}^n)$: the vector space of polynomials,

$f(x) = f(x_1, \ldots, x_n)$, over $\mathbb{R}$.

$\mathrm{Hom}_i(\mathbb{R}^n)$: the subspace consisting of

homogeneous polynomials of degree i.

$\mathcal{P}_l(\mathbb{R}^n) := \oplus_{i=0}^{l} \mathrm{Hom}_i(\mathbb{R}^n)$.

$\mathcal{H}(\mathbb{R}^n)$: the subspace consisting of all the

harmonic polynomials.

$\mathrm{Harm}_l(\mathbb{R}^n) := \mathcal{H}(\mathbb{R}^n) \cap \mathrm{Hom}_l(\mathbb{R}^n)$.

When we consider polynomials on a subset $X \subset \mathbb{R}^n$ we use the following notation.

$\mathcal{P}(X), \mathrm{Hom}_l(X), \mathcal{P}_l(X), \mathcal{H}(X), \mathrm{Harm}_l(X)$

Notation

$\mathbb{R}^n \supset Y$: finite set: $\{\|y\| \mid y \in Y\} = \{r_1, \ldots, r_p\}$,

Possibly one of r_i is 0.

$S_i = \{x \in \mathbb{R}^n \mid \|y\| = r_i\}$,

$Y_i = S_i \cap Y \ (1 \le i \le p)$

$S = \cup_{i=1}^p S_i$, S is called the support of Y

Y is supported by p concentric spheres

$w : Y \longrightarrow \mathbb{R}_{>0}$, a weight function

$w(Y_i) = \sum_{y \in Y_i} w(y)$,

$|S^{n-1}| = \int_{S^{n-1}} d\sigma(x)$, $|S_i| = \int_{S_i} d\sigma_i(x)$,

If $r_i = 0$, then $\frac{1}{|S_i|} \int_{S_i} f(x) d\sigma_i(x) = f(0)$ for $^\forall f(x) \in \mathcal{P}(\mathbb{R}^n)$,

$|S_i| = {r_i}^{n-1} |S^{n-1}|$ for $r_i > 0$.

Euclidean designs

Definition(Neumaier-Seidel, 1988 [24])
(Y, w) is a Euclidean t-design if

$$\sum_{i=1}^{p} \frac{w(Y_i)}{|S_i|} \int_{S_i} f(y) d\sigma_i(y) = \sum_{y \in Y} w(y) f(y)$$

for any polynomial $f(y)$ of degree at most t,
where $w(Y_i) = \sum_{y \in Y_i} w(y)$.

Remarks:

- $p = 1$, $Y \neq \{0\}$, $w(y) \equiv 1 \implies$ Spherical t-designs.
- Assume $0 \notin Y$.
  Then (Y, w) is a Euclidean t-design if and only if $(Y \cup \{0\}, w)$ is a Euclidean t-design ($w(0)$ can be any positive real number).

Natural lower bounds

Theorem (Möller 1979) [23]

(Original theorem was given in terms of general cubature formula)

$\mathbb{R}^n \supset Y$: a finite set, with the support $S = S_1 \cup \cdots \cup S_p$

(1) Y: Euclidean $2e$-design $\Longrightarrow |Y| \geq \dim(\mathcal{P}_e(S))$

(2) Y: Euclidean $(2e+1)$-design

(a) e odd, or e even and $0 \notin Y$

$\Longrightarrow |Y| \geq 2\dim(\mathcal{P}_e^*(S))$

(b) e even and $0 \in Y$

$\Longrightarrow |Y| \geq 2\dim(\mathcal{P}_e^*(S)) - 1$

$\mathcal{P}_e(\mathbb{R}^n) = \oplus_{i=0}^{e}\mathrm{Hom}_i(\mathbb{R}^n)$, $\mathcal{P}_e^*(\mathbb{R}^n) = \oplus_{i=0}^{[\frac{e}{2}]}\mathrm{Hom}_{e-2i}(\mathbb{R}^n)$

Definition of Tight designs

If " = " holds then (Y, w) is called a tight t-design
on p concentric spheres in $\mathbb{R}^n$
Moreover if
(1) $\dim(\mathcal{P}_e(S)) = \dim(\mathcal{P}_e(\mathbb{R}^n))$ (for $t = 2e$),
or
(2) $\dim(\mathcal{P}_e^*(S)) = \dim(\mathcal{P}_e^*(\mathbb{R}^n))$ (for $t = 2e + 1$)
holds, then (Y, w) is called a tight t-design of $\mathbb{R}^n$

If $p \geq [\frac{e+\varepsilon_S}{2}] + 1$ or $p \geq [\frac{e}{2}] + 1$, then (1) and (2) (resp.) are always satisfied. ($\varepsilon_S = 0$ if $0 \notin S$, $\varepsilon_S = 1$ if $0 \in S$)

Formulas for $\dim(\mathcal{P}_e(S))$, $\dim(\mathcal{P}_e^*(S))$ are explicitly known.

$$\dim(\mathcal{P}_i(\mathbb{R}^n)) = \binom{n+e-i-1}{e-i},$$

$$\dim(\mathcal{P}_e(\mathbb{R}^n)) = \binom{n+e}{e} = \sum_{i=0}^{e} \binom{n+e-i-1}{e-i},$$

$$\dim(\mathcal{P}_e^*(\mathbb{R}^n)) = \sum_{i=0}^{[\frac{e}{2}]} \binom{n+e-1-2i}{e-2i}.$$

The explicit formula for $\dim(\mathcal{P}_e(S))$ is known and it depends on the number p of spheres supporting Y (see [20, 15]).
Let $\varepsilon_S = 1$ if $0 \in S$, and $\varepsilon_S = 0$ if $0 \notin S$. Then

$$\dim(\mathcal{P}_e(S)) = \varepsilon_S + \sum_{i=0}^{2(p-\varepsilon_S)-1} \binom{n+e-i-1}{e-i} < \dim(\mathcal{P}_e(\mathbb{R}^n)),$$

for $p \le [\frac{e+\varepsilon_S}{2}]$.

$$\dim(\mathcal{P}_e(S)) = \sum_{i=0}^{e} \binom{n+e-i-1}{e-i} = \dim(\mathcal{P}_e(\mathbb{R}^n)),$$

for $p \ge [\frac{e+\varepsilon_S}{2}] + 1$.

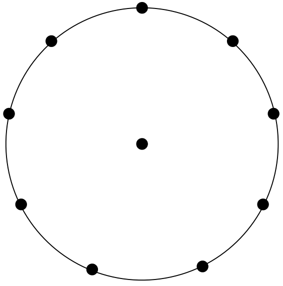
Therefore, in particular for $t = 2e$, $0 \notin Y$ and $p \le [\frac{e}{2}]$, we can express the lower bound of the cardinality of a Euclidean $2e$-design as
$|Y| \ge h_e + h_{e-1} + \ldots + h_{e-p+1}$, where $h_i = \dim(Hom_i(\mathbb{R}^n))$

Euclidean 8-designs in $\mathbb{R}^2$

regular 9-gon is a tight spherical 8-design
regular 7-gon is a tight spherical 6-design

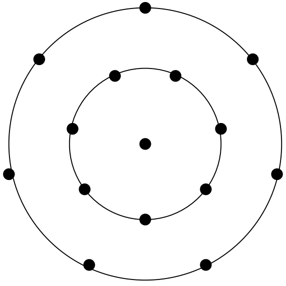
case $\mathbf{0} \in Y$

(regular 9-gon)$\cup\{0\}$



tight 8-design on 2
concentric spheres in $\mathbb{R}^2$

(2 regular 7-gons)$\cup\{0\}$

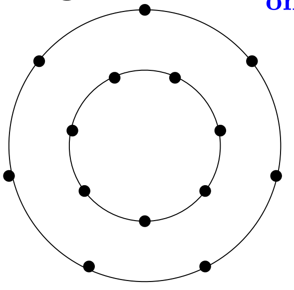


tight 8-design of $\mathbb{R}^2$

$$\dim(\mathcal{P}_4(\mathbb{R}^2)) = 15$$

2 consecutive polygons are at the position obtained by rotating one of them $1/2\times$central angle of the other

case $\mathbf{0} \notin Y$
2 regular 7-gons



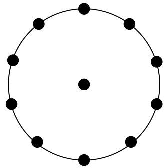
tight 8-design on 2
concentric spheres in $\mathbb{R}^2$

Ratio of the radii r_i/r_{i+1} can be any real number $\neq 1$, and the weight is constant on each circle, and the ratio of the weights w_i/w_1 are determind explicitly by r_i.
$r_1 < r_2 < \cdots < r_p$.

If $t = 9$, then we have the following

Case $0 \in X$. **regular 10-gon is a tight spherical 9-design**

regular 10-gon and 0



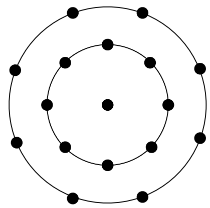
$t = 9$ ($k = 2$), $p = 2$

$|X| = 11$

tight 9-design on 2 concentric spheres

regular 8-gon is a tight spherical 7-design

regular 8-gons and 0



$t = 9$ ($k = 2$), $p = k + 1 = 3$

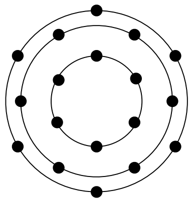
$|X| = 17$

tight 9 design on 3 spheres

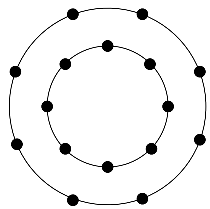
tight 9 design of $\mathbb{R}^2$

Case $0 \notin X$. **regular hexagon is a tight spherical 5-design**

hexagons
$t = 9$,
$p = k + 1 = 3$
$|X| = 18$



tight 9-design of $\mathbb{R}^2$



tight 9 design on 2 concentric spheres
regular 8-gons
$t = 9$, $p = 2$
$|X| = 16$

Ratio of the radii can be any real number $\neq 1$, and the weight is constant on each circle and the ratio of the weights are determind explicitly by the radii.

Note that $\dim(\mathcal{P}_4^*(\mathbb{R}^2)) = 9$.

Existence Theorem

(Seymour and Zaslavsky (1984)[26])
If the N is sufficiently large natural integer, then there always exists a Euclidean t-design Y satisfying $|Y| = N$. (The lower bound of N depends on n and t).

Our interest is finding or classifying tight Euclidean t-design, or Euclidean t-design Y, with smallest possible cardinality.

Euclidean designs and coherent configurations

(Y, w): Euclidean t-design on p concentric spheres.
$Y = \cup_{i=1}^{p} Y_i$.

Notation
$A(Y_i, Y_j) := \{\frac{x \cdot y}{\|x\| \|y\|} \mid x \in Y_i,\ y \in Y_j,\ x \neq y\}$,
$s_{i,j} := |A(Y_i, Y_j)|$, $(s_{i,j} = s_{j,i})$.
(Y_i is a $s_{i,i}$-distance set.)
$A(Y_i, Y_j) = \{\alpha_{i,j}^{(\nu)} \mid 1 \leq \nu \leq s_{i,j}\}$.
$\alpha_{i,i}^{(0)} = 1$, $1 \leq i \leq p$

The following facts are known for tight Euclidean t-designs on p concentric spheres.

$\underline{t = 2e}$

Theorem (B-B 2006 ([3]))
Y: tight $2e$-design on p concentric spheres
$\Longrightarrow$
(1) w is constant on each shell Y_i.
(2) $s_{i,j} \leq e$ $(1 \leq i \leq p)$,
in particular, Y_i is at most an e-distance set.

When t is odd, the situation is a bit complicated.

Theorem (Möller 1979 [23], B-B-Hirao-Sawa 2010 ([10]))
Y: tight $2e+1$-design on p concentric spheres
$\Longrightarrow$
(1) If e is odd, then Y is antipodal and $0 \notin Y$.
Moreover $w(-y) = w(y)$ for any $y \in Y$ (centrally symmetric).
(2) If e is even and $0 \in Y$, then Y is antipodal and
w is centrally symmetric.
(3) If e is even, $0 \notin Y$, and $p \leq \frac{e}{2} + 1$, then Y is antipodal and
w is centrally symmetric.

Theorem (et-B 2006 [15])
Let Y be an antipodal Euclidean tight $(2e+1)$-design.
Assume w is centrally symmetric.
Let $Y = Y^* \cup (-Y^*), Y^* \cap (-Y^*) = \emptyset$ or $\{0\}$.
Then the followng hold:
(1) w is constant on each shell Y_i.
(2) Each $Y_i^* = Y_i \cap Y^*$ is an at most e-distance set.
(3) $s_{i,j} \leq e+1,\ 1 \leq i,j \leq p$ in particular each Y_i is at most an $(e+1)$-distance set.
(4) If w is constant on $Y \backslash \{0\}$, then $p - \varepsilon_S \leq e$.

Theorem (B-B 2010 ([6]))
(1) Y: t-design.
Assume
$w(y) \equiv w_\nu$, $y \in Y_\nu$ $(1 \leq \nu \leq p)$,
$s_{\lambda,\nu} + s_{\nu,\mu} \leq t - 2(p-2)$ for any λ, ν, μ $(1 \leq \lambda, \nu, \mu \leq p)$.
$\Longrightarrow$ Y has the structure of a coherent configuration.

(2) Y antipodal t-design.
Assume
$w(y) \equiv w_\nu$, $x \in X_\nu$ $(1 \leq \nu \leq p)$.
$s_{\lambda,\nu} + s_{\nu,\mu} - \delta_{\lambda,\nu} - \delta_{\nu,\mu} \leq t - 2(p-2)$, for any
λ, ν, μ $(1 \leq \lambda, \nu, \mu \leq p)$.
$\Longrightarrow$ Y has the structure of a coherent configuration..
If $p = 2$ and X_1, $X_2 \neq \{0\}$, then these conditions are satisfied.

Known results for the classification of tight Euclidean t-designs

- $n = 2$:

Verlinden-Cools (1992)
Bajnok (2006)
B-B-Hirao-Sawa (2010)

$\}$ $\leftarrow$ those with $p \leq [\frac{t}{4}] + 1$ are completely described

- $t = 2$:

  B-B-Suprijanto (2007, Europ. J. Comb)

  Y: 1-innerproduct set with a negative inner product, $|Y| = n + 1$

- $t = 3$:

  Bajnok (2006)[1], etB (2005) [15]

  $Y = \{\pm r_i e_i \mid i = 1, \ldots, n\}$, $w(r_i e_i) = \frac{1}{n r_i^2}$,

  $\{e_1, \ldots, e_n\}$ is a canonical basis of $\mathbb{R}^n$.

- We cannot expect the complete classification for $t \geq 4$ in general.

  If p is not small enough, many deformations (non-rigidity) are usually possible (B-B-Suprijanto, 2007)

So, here we mainly study the cases where $t \geq 4$ and $p = 2$.

Known results (continued)

For odd t we have the following:

- $t = 5,\ p = 2:$ etB (2006) [15].

  $Y = \{0\} \cup Y_1$, Y_1 is a spherical tight 5-design,
  If $0 \notin Y$, then Y is similar to one the 4 cases in $\mathbb{R}^n$,
  $n = 2, 3, 5, 6$.
- $t = 7,\ p = 2:$ B-B (2009) [4]

  similar to one of the 3 cases in $\mathbb{R}^n$ $n = 2, 4, 7$.
- $t = 9, p = 2:$ B-B (2011) [5]), non-existence for $n \geq 3$.
- $t \geq 11, p = 2:$ classification is still open for $n \geq 3$.

- $t = 2e + 1 \geq 13, p = 2:$ B-B toappear in [7] (2014)

  n is bounded above by a certain function of t. This means for $n \geq 3$, there are finitely many t-designs for each odd $t \geq 13$.

For even t we have the following:

- $t = 4,\ p = 2$: etB [16] (2009), several interesting examples

  for $n = 2, 4, 5, 6$ and 22.

  For $n = 22$, examples related to tight 4-$(23, 7, 1)$ design in $J(23, 7)$ and tight 4-design in $H(11, 3)$.

  Also partial classification.

  B-B [6] (2010) further partial classification

- $t = 6,\ p = 2$: B-B-Shigezumi [12] (2012),

  one interesting example with $n = 22$ and $|X| = 275$

  McL/$U_4(3)$ +2025(McL/M_{22})

For $p \geq 3$ (and $t \geq 4$), some sporadic examples are known.

- $p = 3,\ t = 7, n = 3,\ |X| = 26$: Bajnok [2] (2007)
- $p = 3,\ t = 5,\ n = 4,\ |X| = 22$: Hirao-Sawa-Zhou [21] (2011)

Classical design theory
(Combinatorial design theory)

$\Downarrow$

Designs in Q-polynomial
association schemes

$\Downarrow$

Spherical designs

$\Downarrow$

Euclidean designs

$\Downarrow$

Relative designs in Q-polynomial
association schemes

Relative t-designs in Q-polynomial association schemes

Some more notation:

$\mathfrak{X} = (X, \{R_i\}_{0 \le i \le d})$: Q-polynomial association scheme.

$\mathcal{F}(X)$: the vector space of all the real valued functions defined on X.

We identify $\mathcal{F}(X)$ and $\mathbb{R}^{|X|}$ and consider $\chi \in \mathcal{F}(X)$ as a vector in $\mathbb{R}^{|X|}$ whose x-entry is defined by $\chi(x)$ for $x \in X$.

For $Y \subset X$, let $\phi_Y \in \mathcal{F}(X)$ be defined by

$$\phi_Y(x) = \begin{cases} 1 & \text{for } x \in Y, \\ 0 & \text{otherwise.} \end{cases} \qquad \big(\text{characteristic function of } Y\big)$$

If $Y = \{u\}$, then we write ϕ_u.

Let $L_i(X)$ be the subspace of $\mathcal{F}(X)$ spanned by all the column vectors of E_i, $0 \le i \le d$. Then we have $\mathcal{F}(X) = L_0(X) \perp L_1(X) \perp \cdots \perp L_d(X)$.

Designs in a Q-polynomial association scheme

Definition(Delsarte 1973, 1977)

t: natural integer

$\chi \in \mathcal{F}(X)$ is a t-design of $\mathfrak{X}$

$\iff$

$E_j\chi = 0$ for $j = 1, 2, \ldots, t$.

The following facts are well known

Let $Y \subset X$.

ϕ_Y is a t-design in Johnson scheme $J(v, k)$

$\iff$ Y is a classical t-(v, k, λ) design in a v point set.

ϕ_Y is a t-designs in Hamming schemes $H(d, q)$

$\iff$ Y is an orthogonal array

Natural lower bound (Delsarte (1973) [17])

$\chi \in \mathcal{F}(X)$: a t-design, $Y := \{y \in X \mid \chi(y) \neq 0\}$,

$$\Longrightarrow |Y| \geq m_0 + m_1 + \cdots + m_e$$

$$\text{where } e = [\tfrac{t}{2}],\ m_i = rank(E_i) = \dim(L_i(X))$$

Compare with the lower bound of Euclidean $2e$-design (mentioned in p. 6) !

Definition (Delsarte (1977) [18])

Let $u_0 \in X$ and $\phi_{u_0} \in \mathcal{F}(X)$ (the characteristic function of u_0).

$\chi \in \mathcal{F}(X)$ is a **relative t-design** with respect to u_0

$$\Longleftrightarrow$$

$E_j\chi$ and $E_j\phi_{u_0}$ are linearly dependent for $j = 1, 2, \ldots t$.

Delsarte(1977) [18]

- $\chi \in \mathcal{F}(X)$ is a t-design
  $\Longrightarrow \chi$ is a relative t-design w.r.t. any u_0 in X
- ϕ_{X_i} is a relative d-design w.r.t. u_0
  for any $i = 0, 1, \ldots, d$

  $$(X_i = \{x \in X \mid (x, u_0) \in R_i\})$$

  ϕ_{X_i} is called a trivial design.

$\mathfrak{X} = H(n,2) = (X, \{R_i\}_{0 \le i \le d})$
$X = F_2^n$, $F_2 = \{0,1\}$, $R_i = \{(x,y) \mid \sharp\{j \mid x_j \neq y_j\} = i\}$,
where $x = (x_1, \ldots, x_n), y = (y_1, \ldots, y_n) \in X$.
Let $u_0 = (0, 0, \ldots, 0)$,
$X_k = \{x \in X \mid (x, u_0) \in R_k\}$.
X_k has the structure of $J(n,k)$ induced by $H(n,2)$.

Delsarte(1977) [18]
Let $Y \subset X_k$. Then the following holds.

Y is a relative t-design w.r.t. u_0
$\Longleftrightarrow$
Y is a t-design of $J(n,k)$

Some more notation:

Canonical inner product:

$f \cdot g = \sum_{x \in X} f(x)g(x),\ f, g \in \mathcal{F}(X)$.

$\{\phi_u \mid u \in X\}$ forms the canonical orthonormal basis of $\mathcal{F}(X)$.

$L_j(X)$:= subspace spanned by $\{E_j\phi_u \mid u \in X\}$.

column space of E_j

Then we have

- $\dim(L_j(X)) = m_j = rank(E_j)$,
- $\mathcal{F}(X) = L_0(X) \perp L_1(X) \perp \cdots \perp L_d(X)$

(with respect to the canonical inner product given above).

Consider (Y, w),

$Y \subset X, \quad Y_r = Y \cap X_r$

w: a positive weight function on Y

Let $p := |\{r \mid Y \cap X_r \neq \emptyset\}|$ and

$\{r_1, r_2, \ldots, r_p\} = \{r \mid Y \cap X_r \neq \emptyset\}$

Let $S = X_{r_1} \cup \cdots \cup X_{r_p}$: support of $Y (= Y_{r_1} \cup \cdots \cup Y_{r_p})$.

Definition B-B (2012) [8](New formulation)
(Y, w):=positive weighted set in a Q-polynomial association scheme $\mathfrak{X}$
$S := X_{r_1} \cup \cdots \cup X_{r_p}$ support of Y. Then
(Y, w) is a relative t-design w.r.t. $u_0 \in X$

$$\Longleftrightarrow$$

$$\sum_{i=1}^{p} \frac{w(Y_{r_i})}{|X_{r_i}|} \sum_{x \in X_{r_i}} f(x) = \sum_{y \in Y} w(y) f(y)$$

for any $f \in L_0(X) + L_1(X) + \cdots + L_t(X)$
Here $w(Y_{r_i}) := \sum_{y \in Y_{r_i}} w(y)$ $(1 \leq i \leq p)$.

Theorem B-B (2012) [8]
(New formulation of Delsarte's idea)
Let χ be a nonnegative function on X, $\overline{\chi} \in \mathcal{F}(X)$ be the function defined by

$\overline{\chi}(x) := \frac{1}{|X_i|} \sum_{y \in X_i} \chi(y)$ for any $x \in X_i$.

Let $Y := \{x \in X \mid \chi(x) \neq 0\}$.

Then the following (1), (2) and (3) are equivalent.

(1) χ is a relative t-design with respect to u_0.

(2) $E_j\chi$ and $E_j\overline{\chi}$ are linearly dependent for any $j = 1,\ 2, \ldots, t$.

(3) Let $w = \chi|_Y$. Then $(Y,\ w)$ is a relative t-design with respect to u_0.

This theorem shows that the original definition of relative t-design by Delsarte and the new formulation given in p.25 are equivalent !

Theorem (B-B (2012) [8])
Let (Y, w) be a relative $2e$-design. Then

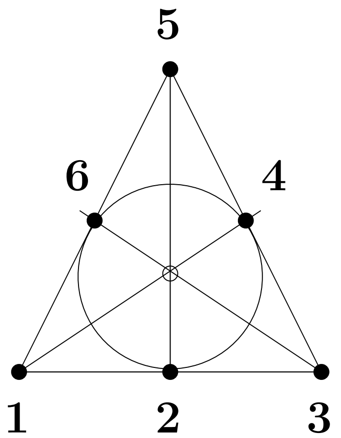
$$|Y| \geq \dim(L_0(S) + L_1(S) + \cdots + L_e(S))$$

holds. Here $S = X_{r_1} \cup X_{r_2} \cup \cdots \cup X_{r_p}$.
(Y, w) is called tight if equality holds in above.

- **The explicit formula for**
  $\dim(L_0(S) + L_1(S) + \cdots + L_e(S))$
  are not known in general.

  It is important to determine the explicit formula.

Examples



relative tight 2-design in $H(6,2)$
w.r.t. $(0,0,0,0,0,0)$

$|Y|=7$

remove one point from
symmetric 2-$(7,3,1)$ design

set of points on a line
considered in F_2^6

e.g. $\{1,2,3\} \Rightarrow (1,1,1,0,0,0) \in X_3$

$\{3,6\} \Rightarrow (0,0,1,0,0,1) \in X_2$

Y consists of 4 blocks with 3 points,
3 blocks with 2 points,

tight 4-$(23,7,1)$ design
$\Longrightarrow$ relative tight 4-design in $H(22,2)$ w.r.t. $(0,0,\ldots,0)$

Known facts related to the relative t-designs of Q-polynomial schemes.

Explicit formula for the lower bound.

It was conjectured that

$$\dim(L_0(S) + L_1(S) + \cdots + L_e)) = m_e + m_{e-1} + \cdots + m_{e-p+1}$$

holds for Q-polynomial schemes and Xiang proved it for the case $H(n, 2)$ (2012) [27].

B-B-Suda-Tanaka (2013) [13], give a condition using the property of Terwilliger algebra of $\mathfrak{X}$ which implies the formula given above. In particular $H(d, q)$ satisfies this condition. See also Li-B-B (2014) [22].

Theorem (B-B-B (2014))
$\mathfrak{X} = (X, \{R_r\}_{0 \le r \le d})$: Q-polynomial scheme.
(Y, w): tight relative $2e$-design on X with respect to $u_0 \in X$.
G: the automorphism group of $\mathfrak{X}$.
Assume that the stabilizer G_{u_0} of u_0 acts transitively on every shell X_r, $1 \le r \le d$.
Then the weight function w is constant on each
$Y_{r_i} = Y \cap X_{r_i}$,
where $\{r_1, \ldots, r_p\} = \{r \mid Y \cap X_r \ne \emptyset\}$

Tight relative 2-designs on $H(n,2)$

Theorem (B-B-B-2014)
Let (Y, w) be a tight relative 2-design of $H(n,2)$ supported by 2 shells, $S = X_{r_1} \cup X_{r_2}$.
Let $N_{r_i} = |Y_{r_i}|$, $w(y) = w_{r_i}$ on $y \in Y_{r_i}$ for $i = 1, 2$.
Then $N_{r_1} + N_{r_2} = n + 1$, and the following (1), (2), (3), and (4) hold.
(1) $2 \leq N_{r_1}, N_{r_2} \leq n - 1$ holds and

$$\frac{w_{r_2}}{w_{r_1}} = \frac{N_{r_1} r_1 (n - N_{r_1})(n - r_1)}{r_2 (N_{r_1} - 1)(n + 1 - N_{r_1})(n - r_2)}.$$

(2) For any integers r_1, r_2 satisfying $1 \leq r_1 < r_2 \leq n-1$, the following holds

$$A(Y_{r_i}) = \left\{ \frac{2(n-r_i)r_i N_{r_i}}{n(N_{r_i}-1)} \right\}, \text{for } i = 1, 2$$

$$A(Y_{r_1}, Y_{r_2}) = \left\{ \frac{n(r_1+r_2) - 2r_1 r_2}{n} \right\}$$

This means that $Y = Y_{r_1} \cup Y_{r_2}$ has a structure of coherent configuration. We also determined existence and nonexistence for all the feasible parameters for $n \leq 30$.

(3) If $n \equiv 6 \pmod 8$, and there exists Hadamard matrix of size $\frac{1}{2}n+1$, then we can construct a tight relative 2 design $Y \subset X_2 \cup X_{\frac{n}{2}}$ ($r_1 = 2$, $r_2 = \frac{n}{2}$) whose weights satisfy $\frac{w_{r_2}}{w_{r_1}} = \frac{8}{n+2}$, i.e., w is not constant on Y.

(4) If $n \leq 30$ and Y is not related to the Hadamard matrices given above, then the weight function is constant on Y.

Recently Hong Yue (student at Hebei Normal Univ.) explicitly determined all the feasible parameters for $31 \leq n \leq 50$ and determined existence and non existence for each of them.
She checked (4) is also true for $31 \leq n \leq 50$.

Except the example given in (3), all the known examples are corresponding to symmetric designs.
The classification problem is still open.

Outline of the the method we use for Q-polynomial scheme $\mathfrak{X}$ in general.

Let (Y, w) be a relative $2e$ design of Q-polynomial scheme $\mathfrak{X}$.
Let $S = X_{r_1} \cup \cdots \cup X_{r_p}$ be the support of Y.
Let $\mathcal{F}(S)$ be the restriction of $\mathcal{F}(X)$ to S.
We consider the inner product on $\mathcal{F}(S)$ defined by

$$\langle f, g \rangle = \sum_{i=1}^{p} \frac{W_{r_i}}{|X_{r_i}|} \sum_{x \in X_{\nu_i}} f(x) g(x)$$

for $f, g \in \mathcal{F}(S)$.
For a Q-polynomial scheme it is known that if $f, g \in L_i(X)$ then $fg \in \sum_{l=0}^{2i} L_{2l}(X)$ holds.

Let $\{\varphi_1, \ldots, \varphi_N\} \subset L_0(X) \perp L_1(X) \perp \cdots \perp L_e(X)$.
Since $\varphi_i\varphi_j \in L_0(X) \perp L_1(X) \perp \cdots \perp L_{2e}(X)$ and we can apply the formula of the definition of relative t-design to $\varphi_i\varphi_j$.
Assume that $\{\varphi_1|_S, \ldots, \varphi_N|_S\}$ is an orthonormal basis of
$L_0(S) + L_1(S) + \cdots + L_e(S)$
with respect to the inner product $\langle\ ,\ \rangle$ given in p.34,
where $L_i(S)$ is the restriction of $L_i(X)$ to S.
Let H be the matrix whose rows are indexed by Y with N columns and (y, i)-entry is defined by $\sqrt{w(y)}\varphi_i(y)$.

Then we have the following

$$({}^t H\ H)(i,j) = \sum_{y \in Y} w(y)\varphi_i(y)\varphi_j(y)$$

$$= \sum_{i=1}^{p} \sum_{x \in X_{r_i}} \frac{W_{r_i}}{|X_{r_i}|}\varphi_i(x)\varphi_j(x) = \delta_{i,j}.$$

Hence we have

$$\mathrm{rank}(H) = |Y| \geq N = \dim(L_0(S) + L_1(S) + \cdots + L_e(S).$$

Assume that (Y, w) is a tight relative $2e$-design.
Then $|Y| = N$ holds. Then H is an invertible matrix and $H\,{}^tH = I$ holds.
Therefore we have

$$(H^tH)(y_1, y_2) = \sum_{i=1}^{N} \sqrt{w(y_1)w(y_2)}\varphi_i(y_1)\varphi_i(y_2) = \delta_{y_1,y_2}.$$

This implies

$$\sum_{i=1}^{N} \varphi_i(x)\varphi_i(y) = \delta(x, y)\frac{1}{w(y)}.$$

If the stabilizer G_{u_0} of u_0 in the automorphism group G of $\mathfrak{X}$ acts transitively on each shell X_r $(1 \leq r \leq d)$, we can prove the following:

For any $\varphi_1, \ldots, \varphi_N \in L_0(X) + L_1(X) + \cdots + L_e(X)$ with the property that $\{\varphi_1|_S, \ldots, \varphi_N|_S\}$ is an orthonormal basis of $L_0(S) + L_1(S) + \cdots + L_e(S)$, the following hold.

$$\sum_{i=1}^{N} \varphi_i(x)^2 = \frac{1}{w_{r_i}}, \quad \text{for any } x \in Y_{r_i},\ i = 1, \ldots, p$$

and

$$\sum_{i=1}^{N} \varphi_i(x)\varphi_i(y) = 0, \qquad \text{for any } x, y \in Y,\ x \neq y.$$

Tight relative 2-designs supported by 2 shells of $J(n, d)$

(The following is done joint with Y. Zhu (student at SJTU) and Eiichi Bannai.)

For the relative 2-design in $J(n, d)$ on 2 shells $X_{r_1} \cup X_{r_2}$, we found out that $(n-1)$ column vectors ϕ_u of E_1, at $u \in X_1$ and the column vector $\phi_0 (\equiv 1)$ of E_0 span $L_0(S) + L_1(S)$, i.e., $\dim(L_0(S) + L_1(S)) = m_1 + m_0 = n$. Starting from these n functions, we compute orthonormal basis of $L_0(S) + L_1(S)$ and determined all the feasible parameters $n, d, r_1, r_2, N_{r_1}, N_{r_2}$ and the relations between the points in Y, for $n \leq 100$.

At this moment the remaining possible parameter up to $n = 100$ is for $n = 16, 36, 45, 64, 96, 100$.
All of them corresponds to the constant weight.
All of the remaining cases have the structure of coherent configurations.
For $n = 16, 36, 45$, Y is 1-distance set and using the symmetric design we actually constructed tight relative 2-designs.

References

[1] B. Bajnok, *On Euclidean designs*, **Adv. Geom. 6, no. 3 (2006), 423–438.**

[2] B. Bajnok, *Orbits of the hyperoctahedral group as Euclidean designs*, **J. Algebraic Combin. 25(4), (2007), 375–397**

[3] E. Bannai and E. Bannai, *On Euclidean tight* 4*-designs*, **J. Math. Soc. Japan, 58 (2006), 775–804.**

[4] E. Bannai and E. Bannai, *Spherical designs and Euclidean designs*, **Recent Developments in Algebra and Related Areas (Beijing, 2007), 1–37, Adv. Lect. Math. 8, Higher Education Press, Beijing; International Press, Boston, 2009.**

[5] E. Bannai and E. Bannai, *Tight 9-designs on two concentric spheres*, **J. Math. Soc. Japan, (64) no.4 (2011), 1359–1376.**

[6] E. Bannai and E. Bannai, *Euclidean designs and coherent configurations*, **Contemporary Mathematics Volume 531, (2010), 59–93.**

[7] E. Bannai and E. Bannai, *Tight t-designs on two concentric spheres*, **Moscow Journal of Combinatorics and Number Theory Volume 4, issue 1 (2014)**

[8] E. Bannai and E. Bannai, *Remarks on the concepts of t-designs*, **J. Appl. Math. Comput., 40 (2012), 195–207. (Proceedings of AGC2010 (13 pages).)**

[9] E. Bannai, E. Bannai, and H. Bannai, *On the existence of tight relative 2-designs on binary Hamming association schemes*, **Discrete Mathematics 314 (2014), 17–37.**

[10] E. Bannai, E. Bannai, M. Hirao, and M. Sawa, *Cubature formulas in numerical analysis and Euclidean tight designs*, **European J. Combin., 31 (2010), 423–441.**

[11] E. Bannai, E. Bannai, M. Hirao, and M. Sawa, *On the existence of minimum cubature formulas for Gaussian measure on $\mathbb{R}^2$ of degree t supported by $[\frac{t}{4}]+1$ circles*,

[12] E. Bannai, E. Bannai, and J. Shigezumi, *A new Euclidean tight 6-design*, **Ann. Comb. 16 (2012), 651 - 659, arXiv:1005.4987 (8 pages).**

[13] E. Bannai, E. Bannai, S. Suda and H. Tanaka, *On relative t-designs in polynomial association schemes*, **arXiv:1303.7163**

[**14**] E. Bannai, E. Bannai and D. Suprijanto, *On the strong non-rigidity of certain tight Euclidean designs*, **European Journal of Combinatorics 28 (2007), 1662–1680.**

[**15**] Et. Bannai, *On antipodal Euclidean tight* $(2e+1)$*-designs*, **J. Algebraic Combin., 24 (2006), 391–414.**

[**16**] Et. Bannai, *New examples of Euclidean tight 4-designs*, **European Journal of Combinatorics 30 (2009), 655–667.**

[**17**] P. Delsarte, *An algebraic approach to the association schemes of the coding theory*, **Thesis, Universite Catholique de Louvain (1973) Philips Res. Repts Suppl. 10 (1973).**

[**18**] P. Delsarte, *Pairs of vectors in the space of an association scheme*, **Philips Res. Repts 32 (1977), 373–411.**

[**19**] P. Delsarte, J. M. Goethals, and J. J. Seidel, *Spherical codes and designs*, **Geom. Dedicata 6 (1977), no. 3, 363–388.**

[**20**] P. Delsarte and J. J. Seidel, *Fisher type inequalities for Euclidean* t*-designs*, **Linear Algebra Appl. 114-115 (1989), 213–230.**

[**21**] M. Hirao, M. Sawa and Y. Zhou, *Some remarks on Euclidean tight designs*, **J. Combin. Theory Ser. A 118, no. 2 (2011), 634–640.**

[**22**] Z. Li, Ei. Bannai, Et. Bannai, *Tight relative 2- and 4-designs on binary Hamming association schemes*, **Graphs and Combin. 30 (2014), 203–227.**

[**23**] H. M. Möller, *Lower bounds for the number of nodes in cubature formulae*, **Numerische Integration (Tagung, Math. Forschungsinst., Oberwolfach, 1978), 221–230, Internat. Ser. Numer. Math., 45, Birkhäuser, Basel-Boston, Mass., 1979.**

[**24**] A. Neumaier and J. J. Seidel, *Discrete measures for spherical designs, eutactic stars and lattices*, **Nederl. Akad. Wetensch. Proc. Ser. A 91=Indag. Math. 50 (1988), 321-334.**

[**25**] H. Yue, *On the existence of tight relative 2-designs on H(n,2) with* $30 < n \leq 50$, **preprint (2014)**

[**26**] P.D. Seymour and T. Zaslavsky, *Averaging sets: A generalization of mean values and spherical designs*, **Adv. Math. 52 (3) (1984), 213–240.**

[**27**] Z. Xiang, *A Fisher type inequality for weighted regular t-wise balanced designs*, **J. Combin. Theory Ser. A 119 (2012) 1523–1527.**

Thank You