A bilinear form relating two Leonard pairs and its applications

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Outline

1. Leonard systems
   - Background
   - Definition of a Leonard system

2. Bilinear form relating two Leonard systems
   - Balanced bilinear form
   - Motivations
   - Parameter array of a Leonard system
   - Results
   - Remarks

Thin irreducible modules and Leonard pairs

- $\Gamma = (X, R)$: a Q-polynomial distance-regular graph
- Fix $x \in X$.
- $T = T(x)$: the Terwilliger algebra with respect to x

Remark

- Each irreducible T-module affords a tridiagonal pair.
- If it is a Leonard pair then the module is said to be thin.

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- the primary T-module : the T-module generated by the characteristic vector of $\{x\}$

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Terminology

- $\mathbb{K}$: a field
- $d \in \mathbb{N}$
- $V := \mathbb{K}^{d+1} \curvearrowleft \mathrm{Mat}_{d+1}(\mathbb{K})$ (from the left)
- $A \in \mathrm{Mat}_{d+1}(\mathbb{K})$: multiplicity-free
  $\overset{\text{def}}{\Longleftrightarrow}$ A has $d+1$ distinct eigenvalues in $\mathbb{K}$

- Suppose A is multiplicity-free.
- $\{\theta_i\}_{i=0}^d \subseteq \mathbb{K}$: the eigenvalues of A
- $\{E_i\}_{i=0}^d \subseteq \mathrm{Mat}_{d+1}(\mathbb{K})$: the primitive idempotents of A; i.e.,

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Five axioms (Terwilliger, 2001)

Definition

A Leonard system in $\mathrm{Mat}_{d+1}(\mathbb{K})$ is a sequence $\Phi = \left(A; A^*; \{E_i\}_{i=0}^{d}; \{E_i^*\}_{i=0}^{d}\right)$ such that:

- A, A^* : multiplicity-free elements in $\mathrm{Mat}_{d+1}(\mathbb{K})$
- $\{E_i\}_{i=0}^{d}$: the primitive idempotents of A
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The pair (A, A^*) is called a Leonard pair.

We use the following notation:

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Definition of a balanced bilinear form

- $\Phi' = \big(A'; A^{*\prime}; \{E'_i\}_{i=0}^{d'}; \{E_i^{*\prime}\}_{i=0}^{d'}\big)$: another Leonard system with $1 \leqslant d' \leqslant d$
- Use $'$ for objects corresponding to Φ' (Example: V', θ'_i, $\theta_i^{*\prime}$)
- $\langle\!\langle , \rangle\!\rangle : V \times V' \longrightarrow \mathbb{K}$: a nonzero bilinear form

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$\langle\!\langle , \rangle\!\rangle$: balanced with respect to Φ, Φ'

$$\overset{\text{def}}{\Longleftrightarrow} \begin{cases} \text{(i)} & \text{There is } \rho\ (0 \leqslant \rho \leqslant d - d') \text{ such that} \\ & \langle\!\langle E_i^* V, E_j^{*\prime} V' \rangle\!\rangle = 0 \text{ if } i - \rho \neq j, \\ \text{(ii)} & \langle\!\langle E_i V, E'_j V' \rangle\!\rangle = 0 \text{ if } i < j \text{ or } i > j + d - d'. \end{cases}$$

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Condition (i)

- There is ρ $(0 \leqslant \rho \leqslant d - d')$ such that $\langle\langle E_i^* V, E_j^{*\prime} V' \rangle\rangle = 0$ if $i - \rho \neq j$:

$$E_0^* V \quad E_1^* V \quad \cdots\cdots\cdots \quad E_{d'}^* V \quad \cdots\cdots \quad E_d^* V$$

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- We call ρ the endpoint of $\langle\langle , \rangle\rangle$.

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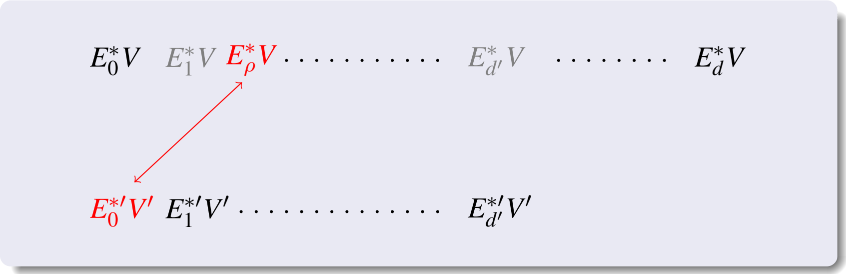

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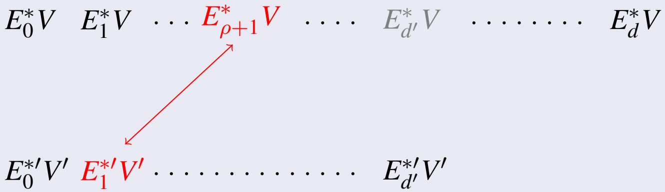

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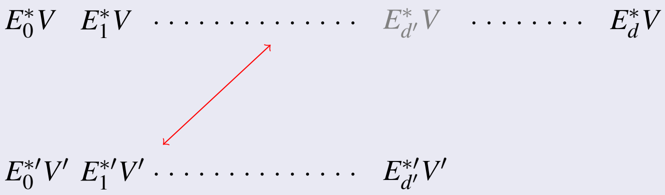

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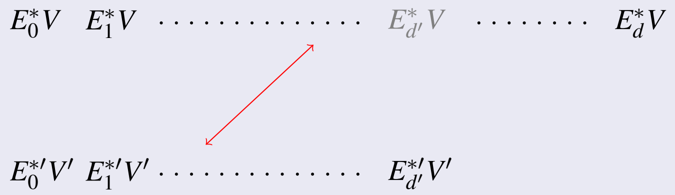

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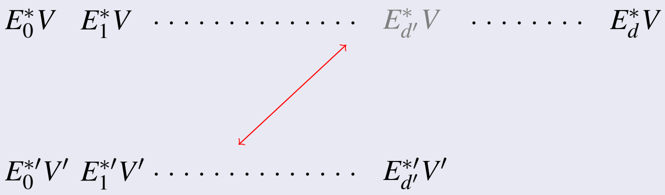

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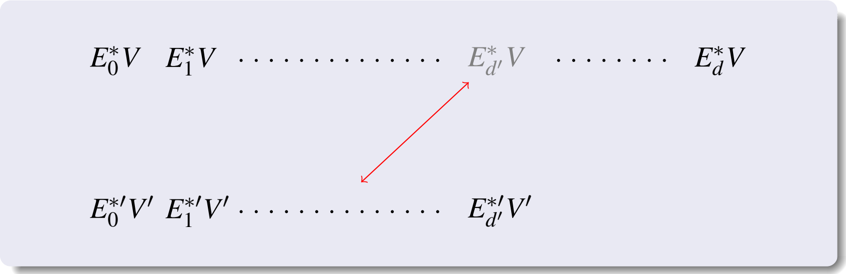

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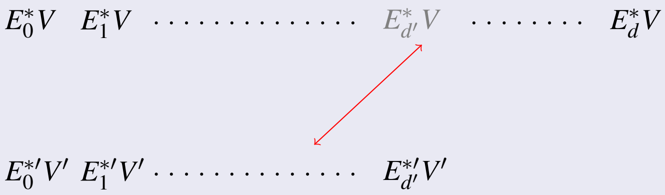

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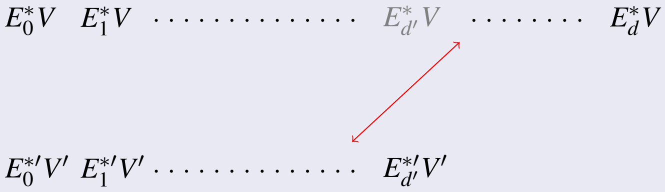

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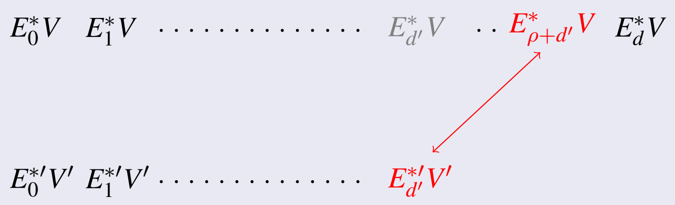

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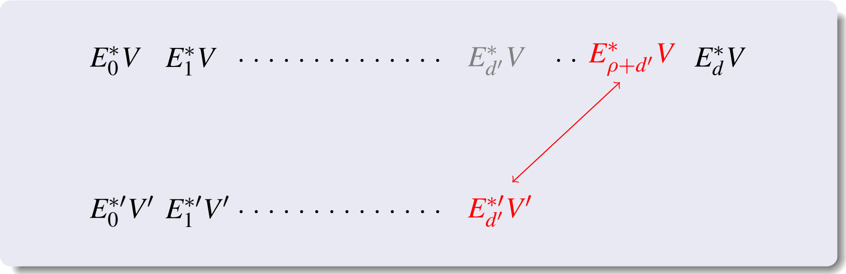

- We call ρ the endpoint of $\langle\langle , \rangle\rangle$.

Condition (ii)

- $\langle\langle E_i V, E'_j V' \rangle\rangle = 0$ if $i < j$ or $i > j + d - d'$:

$$E_0 V \quad E_1 V \quad \cdots\cdots\cdots\cdots\cdots \quad E_{d'} V \quad \cdots\cdots\cdots \quad E_d V$$

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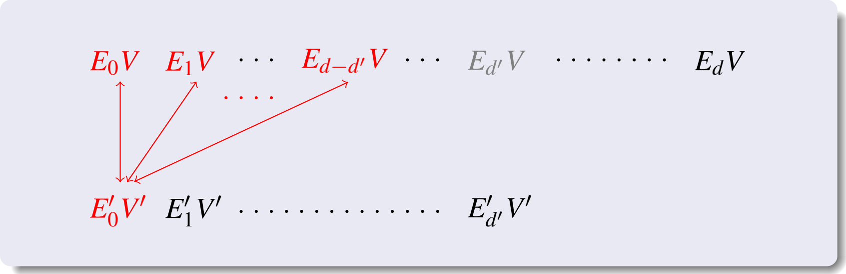

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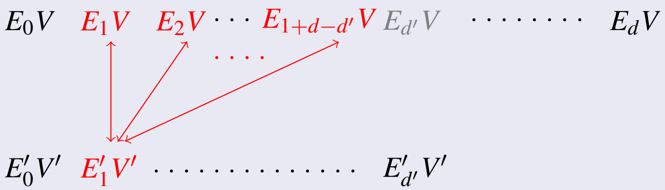

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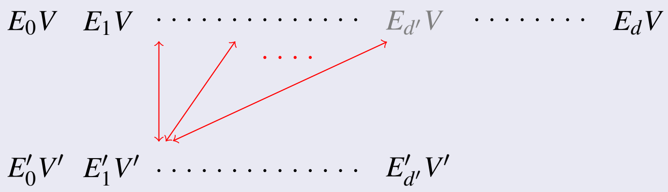

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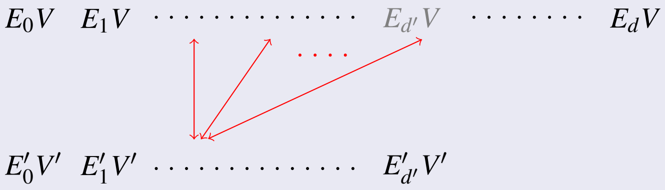

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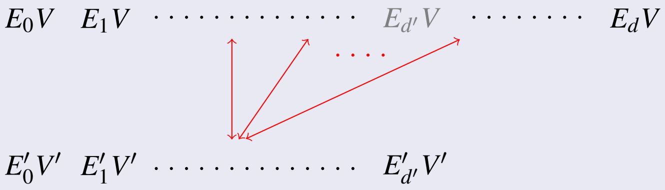

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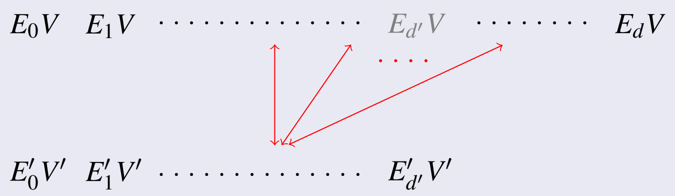

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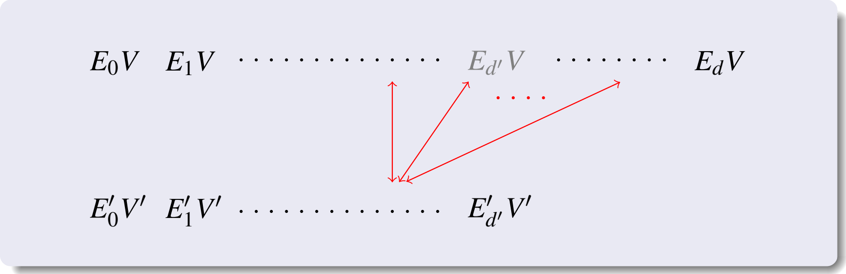

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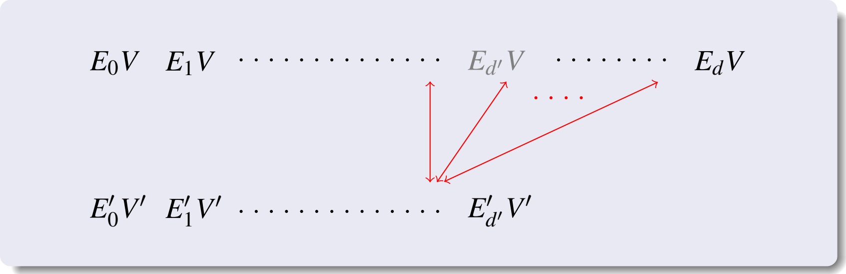

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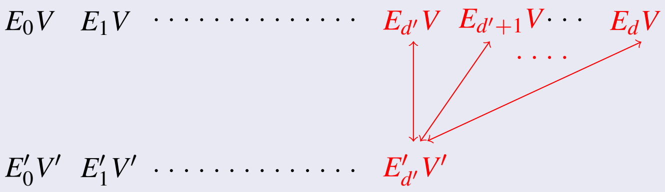

Motivation 1: Subsets with $w + w^* = d$

- $\Gamma = (X, R)$: a Q-polynomial distance-regular graph with diameter d
- C : a proper subset of X
- Brouwer et al. introduced the width w, dual width w^* of C.

Theorem (Brouwer-Godsil-Koolen-Martin, 2003)

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Suppose $w + w^* = d$. Then:

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Example

- $\mathrm{J}(v, d)$: the Johnson graph

$$\mathrm{J}(v,d) \supseteq \mathrm{J}(v-1,d-1) \supseteq \mathrm{J}(v-2,d-2) \supseteq \cdots$$

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Remark

Subsets with $w + w^* = d$ have been applied to:

- Erdős–Ko–Rado Theorem (extremal set theory; 2006)
- Assmus–Mattson Theorem (coding theory; to appear)

Motivation 1: Subsets with $w + w^* = d$ (continued)

- C : connected, $w + w^* = d$
- Fix $x \in C$.
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- $\Gamma = (X, R)$: a Q-polynomial distance-regular graph
- Fix $x, y \in X$ $(x \neq y)$.
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- Terwilliger (1993) studied how thin irreducible modules for T and T' are related.

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Theorem (Suzuki, 2005)

If $2\nu + d' = d$ then W' is thin.

- Suppose $2\nu + d' = d$ and $\partial_\Gamma(x, y) = \nu$.
- W : the primary module for T

If W, W' are not orthogonal then the standard inner product on $\mathbb{C}^X$ is balanced with respect to W, W' (with $\mathbb{K} = \mathbb{C}$, $\rho = \nu^*$).

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What is the parameter array?

The parameter array of Φ is a sequence of the form

$$p(\Phi) = \left(\{\theta_i\}_{i=0}^d; \{\theta_i^*\}_{i=0}^d; \{\varphi_i\}_{i=1}^d; \{\phi_i\}_{i=1}^d\right).$$

- $\{\theta_i\}_{i=0}^d \subseteq \mathbb{K}$: the eigenvalues of A
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Main result

Theorem

There is $\langle\langle , \rangle\rangle : V \times V' \longrightarrow \mathbb{K}$ *which is balanced with respect to* Φ, Φ' *and with endpoint* ρ *if and only if* (i), (ii) *hold below:*

(i) *There are* $\xi^*, \zeta^* \in \mathbb{K}$ *such that*

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Generic case; q-Racah

The most general form of the parameter array is as follows:

$p(\Phi) = p(q, r_1, r_2, s, s^*, d)$ where $r_1 r_2 = s s^* q^{d+1} \neq 0$,

$$\theta_i = \theta_0 + h(1-q^i)(1-sq^{i+1})q^{-i},$$
$$\theta^*_i = \theta^*_0 + h^*(1-q^i)(1-s^*q^{i+1})q^{-i}$$

for $0 \leqslant i \leqslant d$,

$$\varphi_i = hh^*q^{1-2i}(1-q^i)(1-q^{i-d-1})(1-r_1q^i)(1-r_2q^i),$$
$$\phi_i = hh^*q^{1-2i}(1-q^i)(1-q^{i-d-1})(r_1-s^*q^i)(r_2-s^*q^i)/s^*$$

for $1 \leqslant i \leqslant d$.

Closed form of the main result (generic case; q-Racah)

- Suppose
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There is $\langle\langle , \rangle\rangle : V \times V' \longrightarrow \mathbb{K}$ which is balanced with respect to Φ, Φ' and with endpoint ρ if and only if

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(Recall $(r_1 q^\rho)(r_2 q^\rho) = (sq^{d-d'})(s^* q^{2\rho}) q^{d'+1}$.)

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Application: Subsets with $w + w^* = d$

- $\Gamma = (X, R)$: a Q-polynomial distance-regular graph with diameter d
- $C \subseteq X : w + w^* = d$

The following information follow from our theory:

1. When does C induce a Q-polynomial distance-regular graph?
2. When is C "Q-polynomial"?
3. When is C convex (i.e., geodetically closed)?

Answer : (Roughly) (1) $q \neq -1$; (2) $q \neq -1$; (3) Γ has classical parameters.

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3. When is C convex (i.e., geodetically closed)?

Answer : (Roughly) (1) $q \neq -1$; (2) $q \neq -1$; (3) Γ has classical parameters.

The END.

Application: Subsets with $w + w^* = d$

- $\Gamma = (X, R)$: a Q-polynomial distance-regular graph with diameter d
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