Association schemes and orthogonal polynomials

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Notation from “Algebraic Combinatorics I” (1984)

- $\mathfrak{X} = (X, \{R_i\}_{i=0}^D)$: a (symmetric) association scheme
- $A_0, A_1, \ldots, A_D$: the adjacency matrices
- $E_0, E_1, \ldots, E_D$: the primitive idempotents
- $P = (p_j(i))_{i,j=0}^D$, $Q = (q_j(i))_{i,j=0}^D$: the eigenmatrices

$$A_j = \sum_{i=0}^{D} p_j(i) E_i, \quad E_j = \frac{1}{|X|} \sum_{i=0}^{D} q_j(i) A_i \quad (0 \leqslant j \leqslant D)$$

- Set $k_j = p_j(0)$, $m_j = q_j(0)$ $(0 \leqslant j \leqslant D)$.

valency multiplicity

- $\dfrac{p_j(i)}{k_j} = \dfrac{q_i(j)}{m_i} \quad (0 \leqslant i, j \leqslant D)$ $\longleftarrow$ Askey–Wilson duality
- $\displaystyle\sum_{i=0}^{D} m_i\, p_j(i)\, p_\ell(i) = \delta_{j,\ell}|X|k_j$ $\longleftarrow$ orthogonality relation
- $\displaystyle\sum_{i=0}^{D} k_i\, q_j(i)\, q_\ell(i) = \delta_{j,\ell}|X|m_j$ $\longleftarrow$ orthogonality relation*

- Suppose $\mathfrak{X}$ is P- & Q-polynomial.
- Set $\theta_i = p_1(i)$, $\theta_i^* = q_1(i)$ $(0 \leqslant i \leqslant D)$.
- $\exists f_j(x), f_j^*(x) \in \mathbb{R}[x]$ s.t. $\deg f_j(x) = \deg f_j^*(x) = j$ and

$$f_j(\theta_i) = \frac{p_j(i)}{k_j}, \quad f_j^*(\theta_i^*) = \frac{q_j(i)}{m_j} \quad (0 \leqslant i, j \leqslant D).$$

1. $f_0(x) = f_0^*(x) = 1$
2. $x f_j(x) = c_j f_{j-1}(x) + a_j f_j(x) + b_j f_{j+1}(x)$ $\longleftarrow$ 3-term recurrence
3. $x f_j^*(x) = c_j^* f_{j-1}^*(x) + a_j^* f_j^*(x) + b_j^* f_{j+1}^*(x)$ $\longleftarrow$ 3-term recurrence*
4. $f_j(\theta_i) = f_i^*(\theta_j^*)$ $\longleftarrow$ Askey–Wilson duality

Constructions (univariate, terminating)

scheme	polynomials	reference
Hamming	Krawtchouk	Delsarte (1973)
Johnson	Hahn, dual Hahn	Delsarte (1973)
Grassmann	q-Hahn, dual q-Hahn	Delsarte (1976)
bilinear forms	affine q-Krawtchouk	Delsarte (1978)
dual polar	q-Krawtchouk, dual q-Krawtchouk	Stanton (1980)
polygons	q-Racah	Bannai–Ito (1984)
Odd	Bannai-Ito	Bannai–Ito (1984)

- Merit: orthogonality follows automatically.
- Demerit: does not cover full parameter range.
  [Example] Bannai-Ito polynomials have 4 parameters r_1, r_2, s, s^* besides D.

Leonard's theorem (1982)

- Let $\{f_j(x)\}_{j=0}^{D}, \{f_j^*(x)\}_{j=0}^{D} \subset \mathbb{R}[x]$, where $D \in \mathbb{N} \cup \{\infty\}$, satisfy
  1. $f_0(x) = f_0^*(x) = 1$
  2. $x f_j(x) = c_j f_{j-1}(x) + a_j f_j(x) + b_j f_{j+1}(x)$
  3. $x f_j^*(x) = c_j^* f_{j-1}^*(x) + a_j^* f_j^*(x) + b_j^* f_{j+1}^*(x)$
  4. $f_j(\theta_i) = f_i^*(\theta_j^*)$ for some $\{\theta_i\}_{i=0}^{D}, \{\theta_i^*\}_{i=0}^{D} \subset \mathbb{R}$
- Then they are the Askey–Wilson polynomials

$$ {}_4\phi_3\left(\left. \begin{matrix} q^{-j}, s^* q^{j+1}, q^{-y}, s q^{y+1} \\ r_1 q,\ r_2 q,\ r_3 q \end{matrix} \right| q; q \right) \quad (0 \leqslant j \leqslant D) $$

in the variable $x = \theta_0 + h(1-q^y)(1-sq^{y+1})q^{-y}$, where

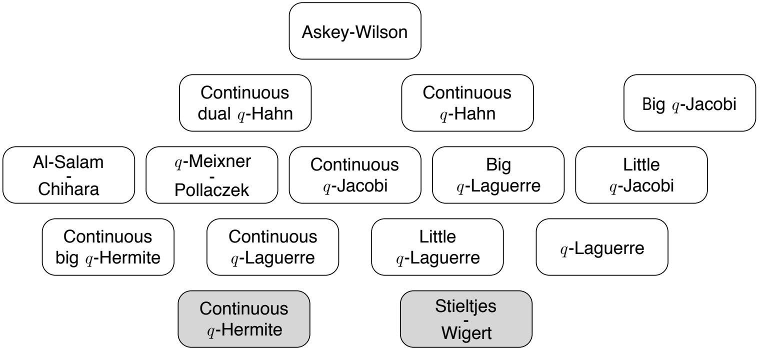
$$ ss^* = r_1 r_2 r_3 \quad (r_3 = q^{-D-1} \text{ if } D < \infty), $$

q-Racah polynomials

or some of their limits in the Askey scheme.

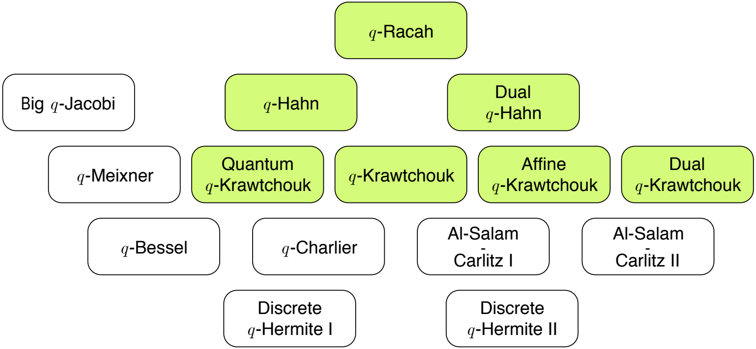
Askey scheme

- q-Hypergeometric orthogonal polynomials, part 1[‡]



‡ taken from: R. Koekoek, P. A. Lesky, and R. F. Swarttouw, Hypergeometric orthogonal polynomials and their q-analogues, Springer-Verlag, Berlin, 2010.

- q-Hypergeometric orthogonal polynomials, part 2‡



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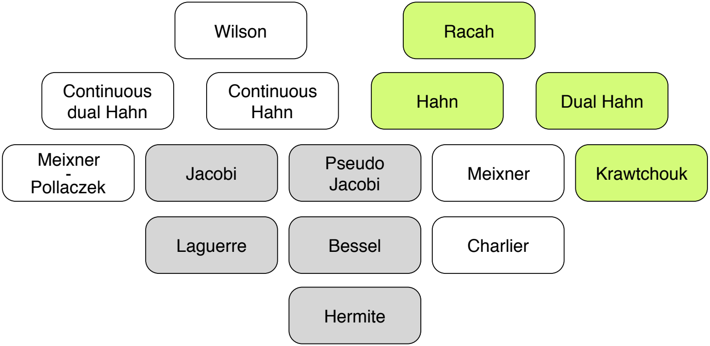
Askey scheme

- Hypergeometric orthogonal polynomials‡



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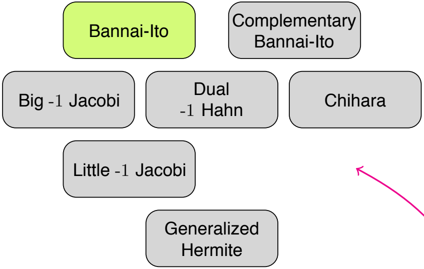
- (-1)-Hypergeometric orthogonal polynomials



[Caution] may be incomplete or wrong!

Remark

The polynomials with $q = -1$ have recently been actively studied by Alexei Zhedanov and others.

Constructions (univariate, non-terminating)

- Suppose $\mathfrak{X}$ is P- & Q-polynomial.
- $\Gamma = (X, R_1)$: distance-regular with valency $k = \theta_0$
- μ_Γ : the (normalized) spectral distribution of Γ on $\mathbb{R}$:

$$\mu_\Gamma\left(\left\{\frac{\theta_i}{\sqrt{k}}\right\}\right) = \frac{m_i}{|X|} \quad (i = 0, 1, \ldots, D).$$

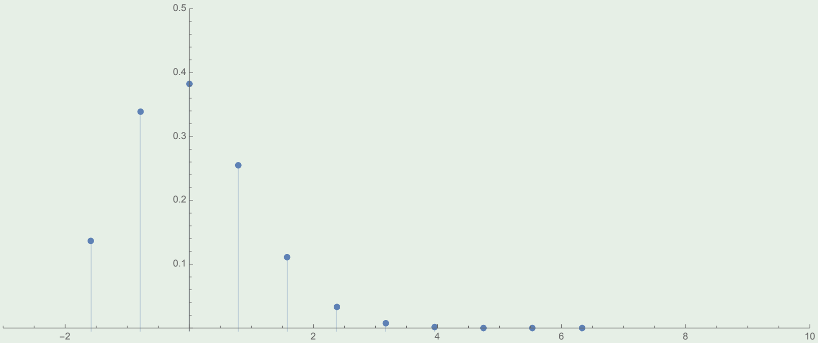
- Hora (1998) described the limit distributions

$$\mu_\Gamma \to \mu_\infty \quad \text{(weakly)}$$

for various growing families of classical examples, including Hamming schemes and Johnson schemes.

Example (Hamming schemes $H(D, q)$)

- $q/D \to q' \in (0, \infty)$



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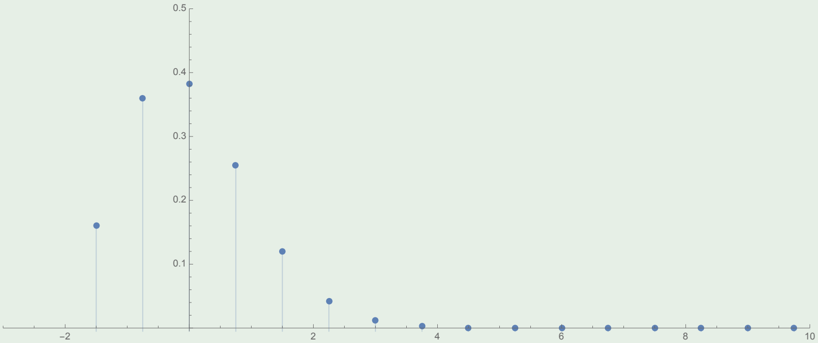

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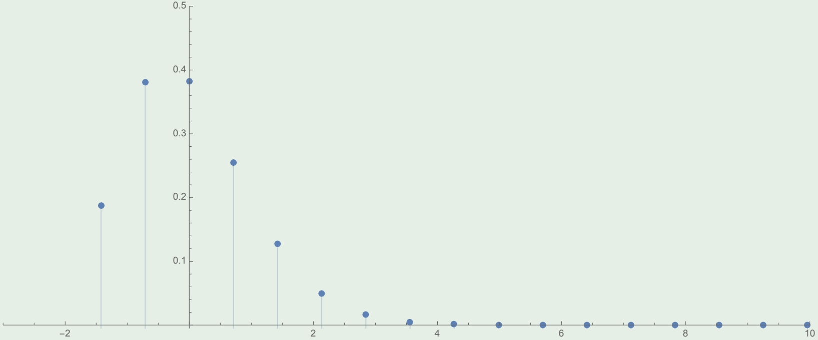

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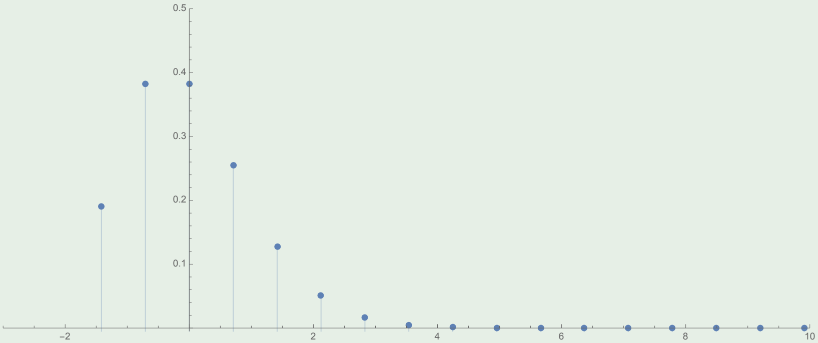
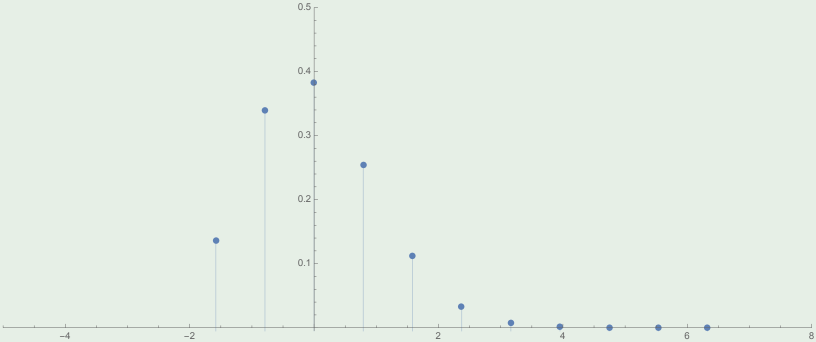


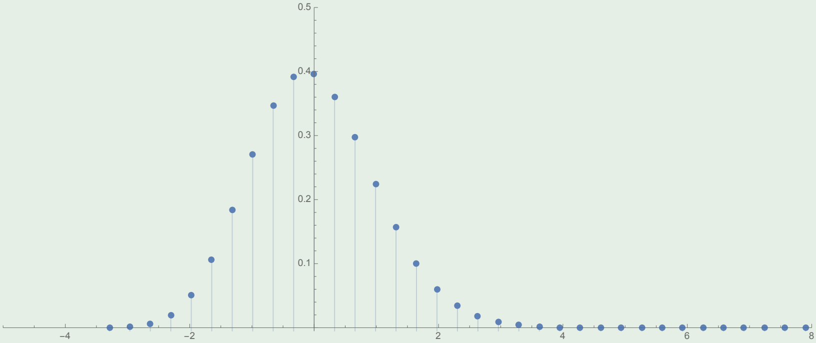
Example (Hamming schemes $H(D,q)$)

- $q/D \to 0$



Example (Hamming schemes $H(D, q)$)

- $q/D \to \textcolor{red}{0}$



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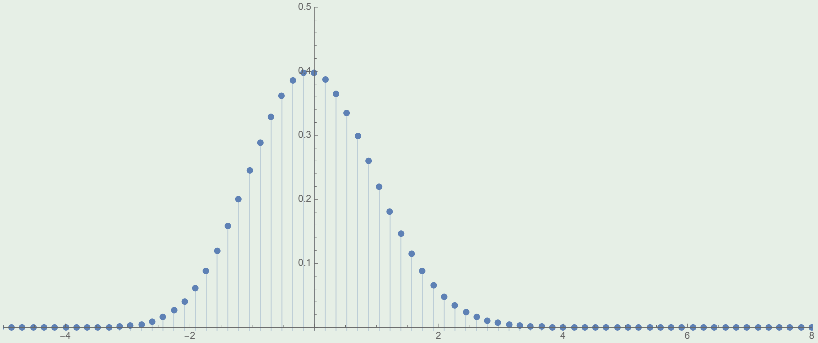

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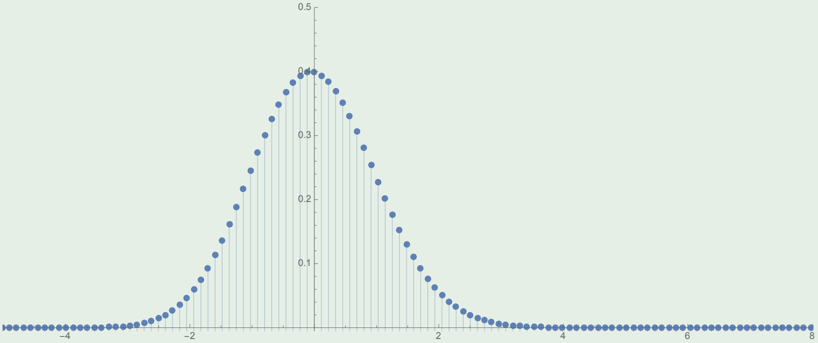


Example (Hamming schemes $H(D,q)$)

- $q/D \to q' \in [0,\infty)$

	distribution	polynomials
original	binomial	Krawtchouk
$q' > 0$	Poisson	Charlier
$q' = 0$	Gaussian	Hermite

- The orthogonality of the limit polynomials is almost automatic.

Example (Johnson schemes $J(N, D)$)

- $N/D \to N' \in [2, \infty)$

	distribution	polynomials
original	(no name?)	dual Hahn
$N' > 2$	geometric	Meixner
$N' = 2$	exponential	Laguerre

- Hora (1998) obtained these results by evaluating the spectra directly, which is usually quite complicated.
- Hora, Obata and others then revisited the results from the viewpoint of the quantum decomposition

$$A_1 = L + F + R$$

in the Terwilliger algebra of $\mathfrak{X}$.
- Their motivation is on finding concrete combinatorial models for quantum probability theory.‡

‡ See: A. Hora and N. Obata, Quantum probability and spectral analysis of graphs, Springer, Berlin, 2007.

Remark

Koornwinder (2011) found a new limit from q-Racah to big q-Jacobi in which the orthogonality property remains present.

Constructions (multivariate, terminating)

- $\mathfrak{X}$: arbitrary
- $\mathfrak{A} = \langle A_0, \ldots, A_D \rangle = \langle E_0, \ldots, E_D \rangle$: the Bose–Mesner algebra

Observation (Delsarte, 1973)

- $\mathrm{Sym}^n(\mathfrak{A})$: the BM algebra of an association scheme $(\forall n \in \mathbb{N})$

extension of length n

Example

- $D = 1 \implies H(n, |X|)$

Theorem (Mizukawa–T. (2004))

- *The eigenmatrices of $\mathrm{Sym}^n(\mathfrak{A})$ are described by the D-variate Krawtchouk polynomials.*

Example ($D = 2$)

- The bivariate Krawtchouk polynomials in the variables x, y are given in terms of the Aomoto–Gelfand hypergeometric series:

$$\sum_{\substack{a,b,c,d\in\mathbb{Z}_{\geqslant 0}\\ a+b+c+d\leqslant n}} \frac{(-i)_{a+b}(-j)_{c+d}(-x)_{a+c}(-y)_{b+d}}{(-n)_{a+b+c+d}\, a!\, b!\, c!\, d!}\, \alpha^a\beta^b\gamma^c\delta^d$$

for $i, j \in \mathbb{Z}_{\geqslant 0}$ with $i + j \leqslant n$, for some fixed $\alpha, \beta, \gamma, \delta \in \mathbb{R}$.

Remark

- There are some other examples which give rise to multivariate orthogonal polynomials, e.g., the non-binary Johnson schemes, but these examples are imprimitive.

defined by Tarnanen–Aaltonen–Goethals (1985),
but studied earlier by Dunkl (1976)

Constructions (multivariate, non-terminating)

- Suppose $D = 2 \implies \mathfrak{X}$: P- & Q-polynomial
- P is of the form

$$P = \begin{pmatrix} 1 & k & \overline{k} \\ 1 & r & -r-1 \\ 1 & s & -s-1 \end{pmatrix},$$

  where $|X| = 1 + k + \overline{k}$ and $k \geqslant r \geqslant 0 > s$.
- $\Gamma = (X, R_1)$, $\overline{\Gamma} = (X, R_2)$: strongly regular

complement of Γ

Remark

- When $D = 1$,

$$\Gamma = (X, R_1) = K_{|X|} \text{ (complete graph).}$$

- The adjacency matrices in $\mathrm{Sym}^n(\mathfrak{A})$ corresponding to the degree 1 Krawtchouk polynomials are

$$\boldsymbol{A} = \sum_{\ell=1}^{n} I \otimes \cdots \otimes I \otimes \underset{\overset{\frown}{\ell}}{A_1} \otimes I \otimes \cdots \otimes I,$$

$$\overline{\boldsymbol{A}} = \sum_{\ell=1}^{n} I \otimes \cdots \otimes I \otimes \underset{\overset{\frown}{\ell}}{A_2} \otimes I \otimes \cdots \otimes I.$$

- $\boldsymbol{A}$ (resp. $\overline{\boldsymbol{A}}$) : the adjacency matrix of Γ^n (resp. $\overline{\Gamma}^n$)

n^{th} Cartesian power of Γ

Remark

- When $D = 1$, $\Gamma^n = (K_{|X|})^n$ is the Hamming graph $H(n, |X|)$.

- The eigenspaces of $\mathrm{Sym}^n(\mathfrak{A})$ are parameterized by

$$\{(i,j) : i,j \in \mathbb{Z}_{\geqslant 0},\ i+j \leqslant n\}.$$

- Consider the joint spectrum of $\Gamma^n, \overline{\Gamma}^n$:

$$\begin{pmatrix} (\theta_{0,0}, \overline{\theta}_{0,0}) & \cdots & (\theta_{i,j}, \overline{\theta}_{i,j}) & \cdots \\ m_{0,0} & \cdots & m_{i,j} & \cdots \end{pmatrix}$$

- $\mu_{\Gamma^n,\overline{\Gamma}^n}$: the (normalized) joint spectral distribution of $\Gamma^n, \overline{\Gamma}^n$ on $\mathbb{R}^2$:

$$\mu_{\Gamma^n,\overline{\Gamma}^n}\left(\left\{\left(\frac{\theta_{i,j}}{\sqrt{\boldsymbol{k}}}, \frac{\overline{\theta}_{i,j}}{\sqrt{\boldsymbol{h}}}\right)\right\}\right) = \frac{m_{i,j}}{|X|^n} \quad (i,j \in \mathbb{Z}_{\geqslant 0},\ i+j \leqslant n),$$

where $\boldsymbol{k} = \theta_{0,0}$ (resp. $\boldsymbol{h} = \overline{\theta}_{0,0}$) is the valency of Γ^n (resp. $\overline{\Gamma}^n$).

- Let

$$k/n \to k', \quad \overline{k}/n \to \overline{k}', \quad r/n \to r', \quad s/n \to s'.$$

valency of Γ valency of $\overline{\Gamma}$

[Note] Γ is not fixed and may vary with n.

Theorem (Morales–Obata–T.)

We have $r' = 0$ or $s' = 0$, and

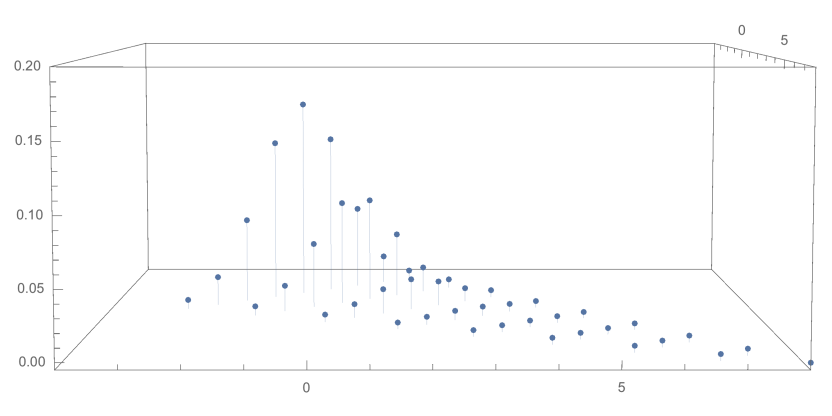
$$\mu_{\Gamma^n, \overline{\Gamma}^n} \to \mu_\infty \quad \textit{(weakly)}$$

where μ_∞ takes one of the following:

1. $k' > 0$, $\overline{k}' = -s' > 0$, $r' = 0$: *bivariate Poisson;*
2. $k' = r' > 0$, $\overline{k}' > 0$, $s' = 0$: *bivariate Poisson;*
3. $k' + \overline{k}' > 0$, $r' = s' = 0$: *product of Poisson and Gaussian;*
4. $k' = \overline{k}' = r' = s' = 0$: *bivariate Gaussian*

Example (Paley graphs $\mathrm{Paley}(q)$)

- $\Gamma = \mathrm{Paley}(q) \quad (q \equiv 1 \bmod 4)$
- $q/n \to q' \in (0, \infty)$



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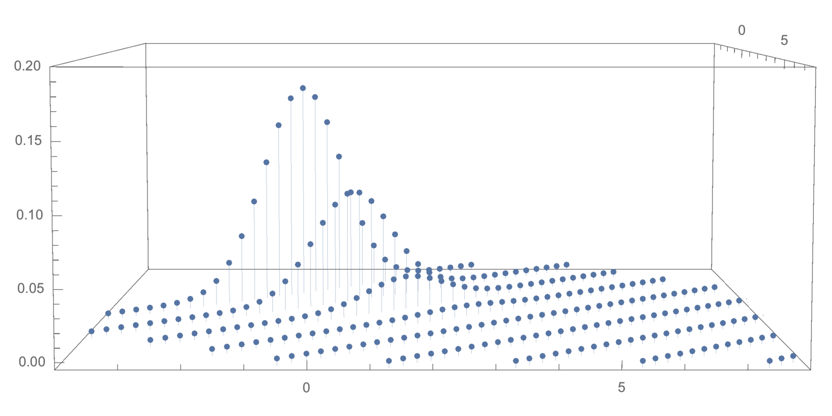

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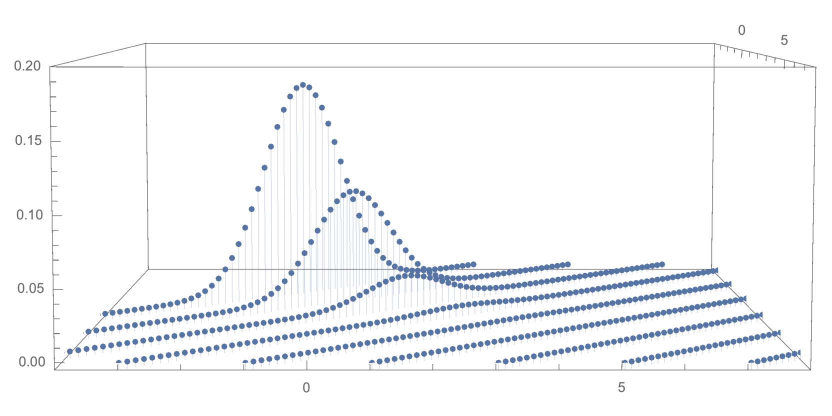

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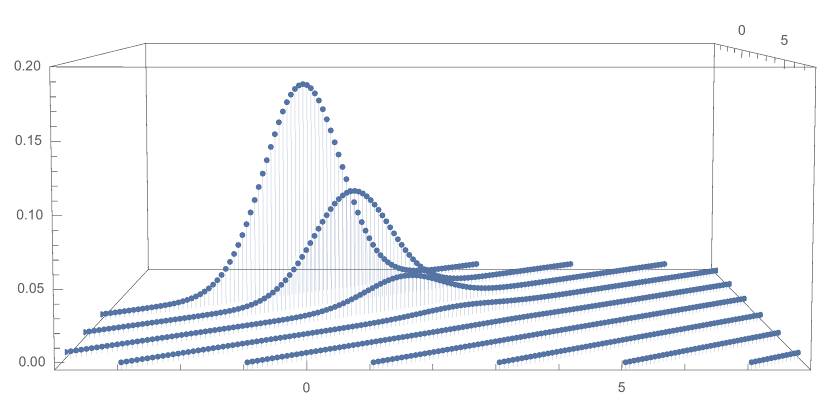
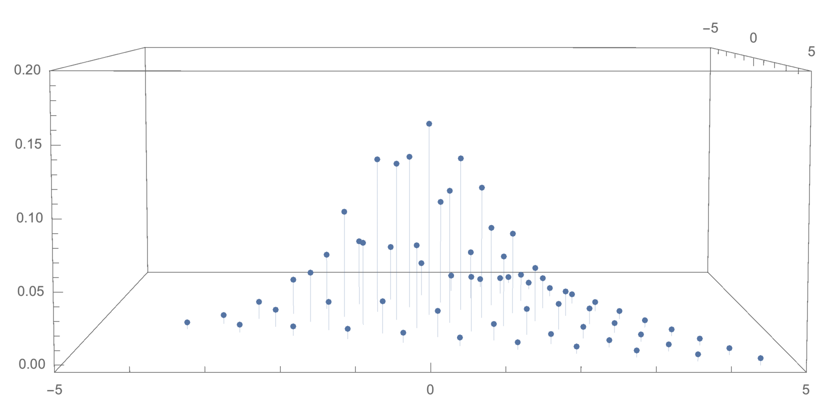


Example (Paley graphs $\mathrm{Paley}(q)$)

- $\Gamma = \mathrm{Paley}(q) \quad (q \equiv 1 \bmod 4)$
- $q/n \to 0$



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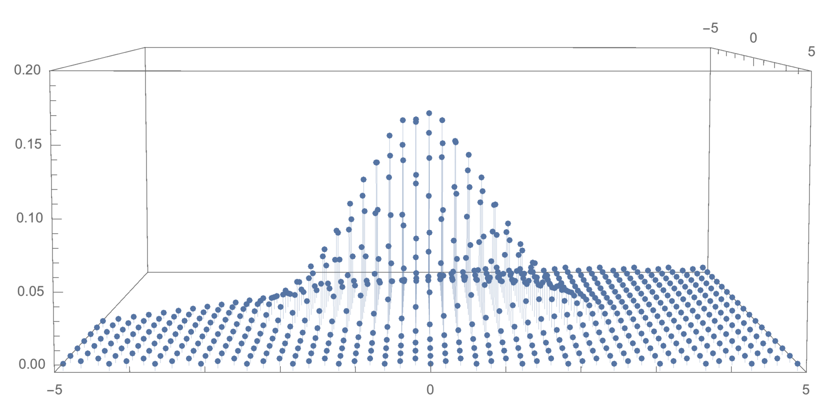

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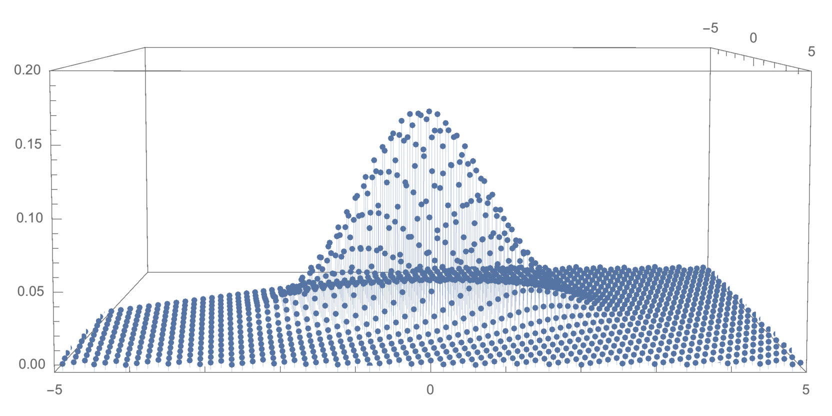

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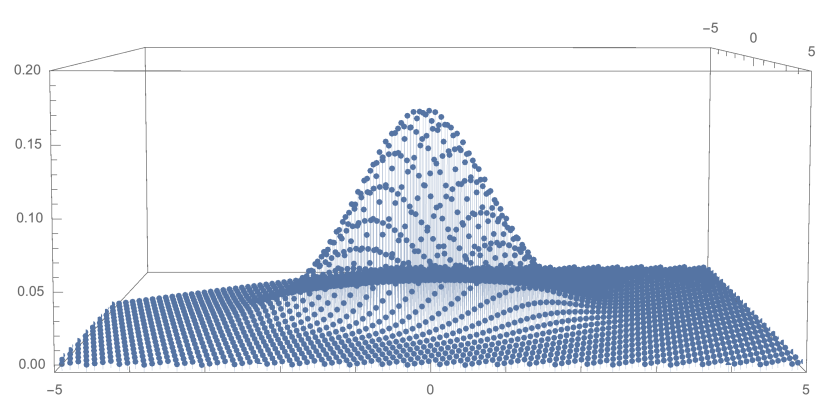


	distribution	polynomials
original	trinomial	bivariate Krawtchouk
1, 2	bivariate Poisson	bivariate Charlier
3	Poisson $\times$ Gaussian	bivariate Charlier–Hermite
4	bivariate Gaussian	bivariate Hermite

seems previously unnoticed

1. $k' > 0$, $\overline{k}' = -s' > 0$, $r' = 0$
2. $k' = r' > 0$, $\overline{k}' > 0$, $s' = 0$
3. $k' + \overline{k}' > 0$, $r' = s' = 0$
4. $k' = \overline{k}' = r' = s' = 0$

- Again, our motivation comes from quantum probability theory.

Constructions (non-symmetric, terminating)

- There is a fundamental connection:

$$\text{AW polynomials} \;\longleftrightarrow\; \text{the DAHA of type } (C_1^\vee, C_1)$$

- Fix $\kappa_0, \kappa_1, \kappa_0', \kappa_1' \in \mathbb{C}\backslash\{0\}$.
- $\mathcal{H} = \mathcal{H}(\kappa_0, \kappa_1, \kappa_0', \kappa_1'; q)$: the DAHA of type $(C_1^\vee, C_1)$

$$\begin{aligned} &\text{generators:} && \mathcal{T}_0^{\pm 1}, \mathcal{T}_1^{\pm 1}, \mathcal{X}^{\pm 1} \\ &\text{relations:} && (\mathcal{T}_i - \kappa_i)(\mathcal{T}_i + \kappa_i^{-1}) = 0 \quad (i = 0, 1) \\ & && (\mathcal{T}_i' - \kappa_i')(\mathcal{T}_i' + \kappa_i'^{-1}) = 0 \quad (i = 0, 1) \end{aligned}$$

where

$$\mathcal{T}_0' := q^{-1/2}\, \mathcal{X} \mathcal{T}_0^{-1}, \quad \mathcal{T}_1' := \mathcal{X}^{-1} \mathcal{T}_1^{-1}.$$

- $f(z) \in \mathbb{C}[z, z^{-1}]$: a Laurent polynomial

$$f : \text{symmetric} \overset{\text{def}}{\iff} f(z) = f(z^{-1})$$
$$f : \text{non-symmetric} \overset{\text{def}}{\iff} f(z) \neq f(z^{-1})$$

- {symmetric Laurent polynomials in z} = {polynomials in $z + z^{-1}$}

- Recall the AW polynomials

$$f_j(x) = {}_4\phi_3\left(\begin{matrix} q^{-j}, s^*q^{j+1}, q^{-y}, sq^{y+1} \\ r_1q,\ r_2q,\ r_3q \end{matrix} \,\middle|\, q;q\right) \qquad (j=0,1,2,\dots),$$

which we view now as symmetric Laurent polynomials in the variable $z = s^{1/2}q^{1/2+y}$:

$$f_j(z) = {}_4\phi_3\left(\begin{matrix} q^{-j}, s^*q^{j+1}, (sq)^{1/2}z^{-1}, (sq)^{1/2}z \\ r_1q,\ r_2q,\ r_3q \end{matrix} \,\middle|\, q;q\right)$$

- There is a faithful irreducible representation

$$\pi : \mathcal{H} \to \mathrm{End}(\mathbb{C}[z, z^{-1}])$$

such that the symmetric AW polynomials are eigenfunctions of

$$\mathcal{X} + \mathcal{X}^{-1}.$$

- $\tilde{f}_j(z)$: the AW polynomials with parameters $\tilde{s}, \tilde{s}^*, \tilde{r}_1, \tilde{r}_2, \tilde{r}_3$, where

$$\tilde{s} = sq^2, \quad \tilde{s}^* = s^* q^2, \quad \tilde{r}_1 = r_1 q, \quad \tilde{r}_2 = r_2 q, \quad \tilde{r}_3 = r_3 q^2.$$

- The non-symmetric AW polynomials

$$e_{\pm j}(z) \in \mathbb{C}[z, z^{-1}] \quad (j = 0, 1, 2, \dots)$$

are eigenfunctions of

$$\mathcal{X}^{\pm 1},$$

and are written as $\mathbb{C}$-linear combinations of

$$f_j(z), \qquad \frac{(1-(sq)^{1/2}z)(1-r_3(q/s)^{1/2}z)}{z}\tilde{f}_{j-1}(z).$$

symmetric

non-symmetric

symmetric

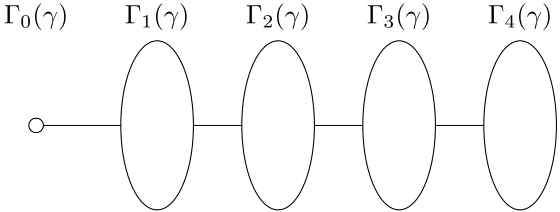
- Suppose $\mathfrak{X}$ is P- & Q-polynomial of q-Racah type
- Suppose further that $\Gamma = (X, R_1)$ has a clique C of size

$$|C| = 1 - \frac{k}{\theta_D} \qquad \text{(Hoffmann bound).}$$

must be the least eigenvalue of Γ

- Fix $\gamma \in C$.

- The distance partition from γ : $\longleftarrow$ equitable



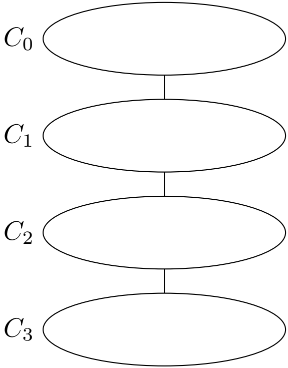
$$\Gamma_i(\gamma) = \{\delta \in X : \partial_\Gamma(\gamma, \delta) = i\}$$

$r_3 = q^{-D-1}$

- This affords the regular representation of $\mathfrak{A}$, and captures the (symmetric) q-Racah polynomials $f_j(z)$.

  [Keywords] primary $T(\gamma)$-module, Leonard pairs

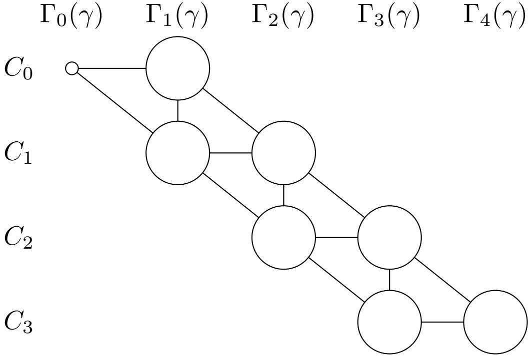
- The distance partition from C : $\longleftarrow$ equitable



$C_i = \{\delta \in X : \partial_\Gamma(C, \delta) = i\}$

- There is associated another set of q-Racah polynomials.

- The distance partition from both γ and C : $\longleftarrow$ equitable



- Lee (2013, 2017) defined an irreducible $\mathcal{H}$-module structure here, and then gave a combinatorial proof of the orthogonality of the non-symmetric q-Racah polynomials.

- Lee and T. (2017+) applied this approach to the dual polar schemes, and obtained the non-symmetric dual q-Krawtchouk polynomials.
- We encounter nil-DAHAs defined by Cherednik and Orr (2015).
- For $(C_1^\vee, C_1)$, the definition reads as follows:

There is a flexibility in the definition.

- Fix $\kappa, \kappa' \in \mathbb{C}\backslash\{0\}$.
- $\overline{\mathcal{H}} = \overline{\mathcal{H}}(\kappa, \kappa')$: a nil-DAHA of type $(C_1^\vee, C_1)$

generators: $\mathcal{T}^{\pm 1}, \mathcal{U}, \mathcal{X}^{\pm 1}$

relations:

$$(\mathcal{T} - \kappa)(\mathcal{T} + \kappa^{-1}) = \mathcal{U}(\mathcal{U} + 1) = 0$$
$$(\mathcal{T}' - \kappa')(\mathcal{T}' + \kappa'^{-1}) = \mathcal{U}'^2 = 0$$

where

$$\mathcal{T}' = \mathcal{X}\mathcal{T}^{-1}, \quad \mathcal{U}' = \mathcal{X}^{-1}(\mathcal{U} + 1).$$