The independence number of the orthogonality graph in dimension 2^k

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About myself

- **Research area:**
  algebraic combinatorics, algebraic graph theory, spectral graph theory, etc.

- **Professional background**
  Mar ’04 : Ph.D. in Math from Kyushu U
  Apr ’04 – Mar ’07 : JSPS postdoc at GSIS, Tohoku U
  Apr ’07 – Sep ’07 : in USA (WPI, MIT)
  Oct ’07 – Jul ’12 : Assist. Prof. at GSIS, Tohoku U
  Aug ’12 – : Assoc. Prof. at GSIS, Tohoku U
  Apr ’17 – : Assoc. Prof. at RACMaS, Tohoku U

Pseudo-telepathy game (Brassard–Cleve–Tapp ('99))

- Alice & Bob have no communication after the game starts.

- They receive n-bit strings (questions)

$$x = (x_1, \ldots, x_n),\ y = (y_1, \ldots, y_n)\ \in \{0,1\}^n$$

where $n = 2^k$, such that

$$\partial(x, y) = 0 \text{ or } n/2.$$

$\#\{i : x_i \neq y_i\}$ (Hamming distance)

- They respond with s-bit strings (answers)

$$a = (a_1, \ldots, a_s),\ b = (b_1, \ldots, b_s)\ \in \{0,1\}^s.$$

- They win if $x = y \iff a = b$.

Pseudo-telepathy game (Brassard–Cleve–Tapp ('99))

Theorem (Brassard–Cleve–Tapp)**.** *The pseudo-telepathy game can be won with*

$$s = k = \log_2 n,$$

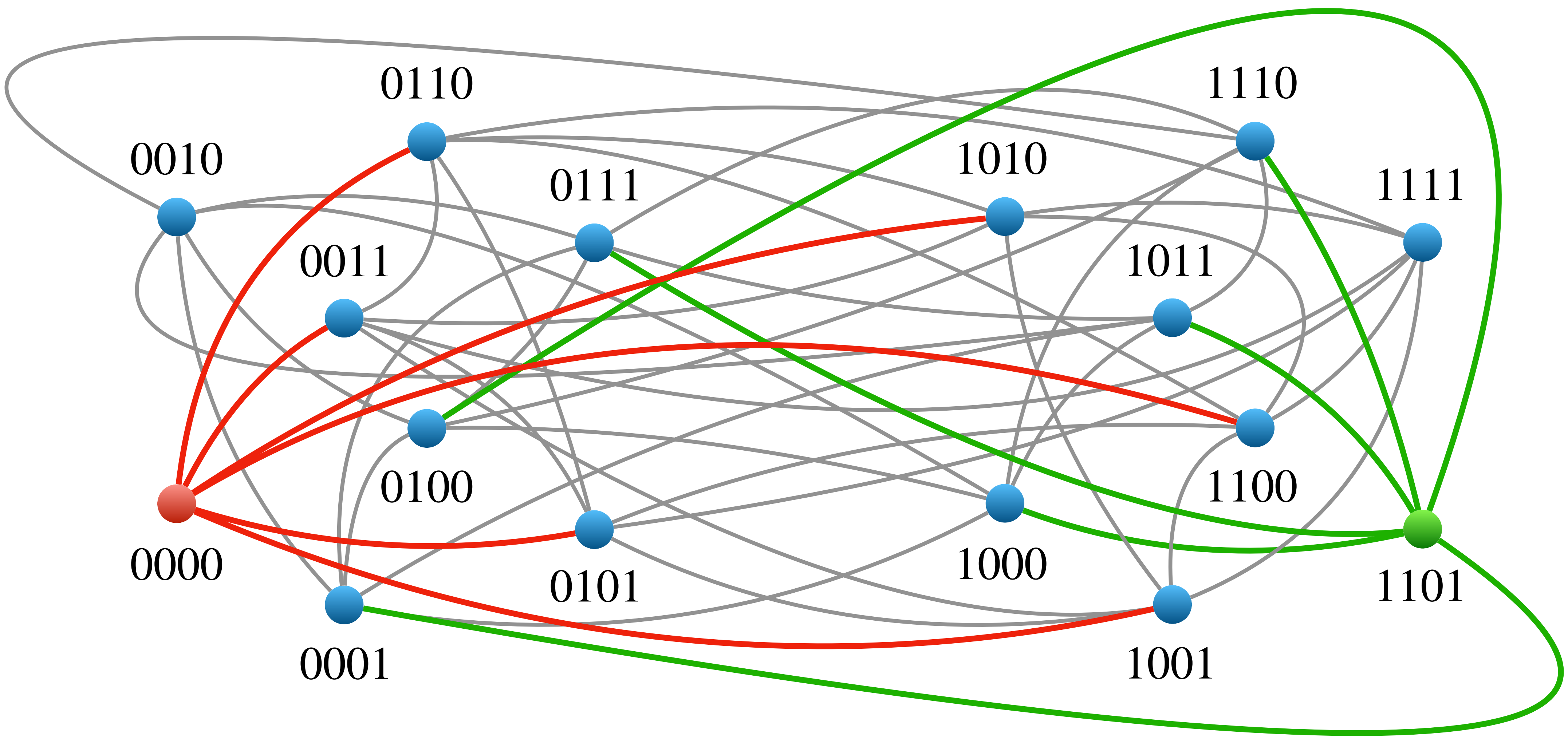
if Alice & Bob are allowed to use k-qubit quantum systems in the maximally entangled state.

Question. How small can s be to win the classical pseudo-telepathy game?

The orthogonality graph Ω_n $(n = 2^k)$

- $V = V(\Omega_n) = \{0,1\}^n$ (vertex set)
- $E = E(\Omega_n) = \left\{ \{x, y\} : x, y \in V,\ \partial(x, y) = n/2 \right\}$ (edge set)

Ω_4



Coloring of Ω_n

- Alice & Bob receive $x,\ y\ \in V = \{0,1\}^n$ such that
$$\partial(x,y) = 0 \text{ or } n/2.$$

- They respond with $a,\ b\ \in \{0,1\}^s$ so that
$$x = y \iff a = b.$$

- Alice & Bob's answers are functions $f : x \mapsto a,\quad g : y \mapsto b$.

- We must have $f = g$.

- Moreover, we also have
$$\partial(x,y) = n/2 \implies f(x) \neq f(y).$$

Coloring of Ω_n

- The function $f : V \to \{0,1\}^s$ satisfies

x  $y \implies f(x) \neq f(y).$

x is "colored" by a

- In other words, for every $a \in \{0,1\}^s$, the set

$$f^{-1}(a) = \left\{x \in V : f(x) = a\right\}$$

is an independent set, i.e., no two vertices are adjacent.

- Moreover, these sets partition the vertex set V:

$$V = \bigsqcup_{a \in \{0,1\}^s} f^{-1}(a).$$

- Thus, Ω_n has a coloring with 2^s colors.

The chromatic number of Ω_n

- The chromatic number $\chi(\Omega_n)$ of Ω_n is the smallest number of colors in a coloring of Ω_n.

Remark. We can show $\chi(\Omega_n) \geqslant n = 2^k$ in general.

Summary.

- Alice & Bob win the classical pseudo-telepathy game
  $\iff \Omega_n$ has a coloring with 2^s colors
  $\iff s \geqslant \log_2 \chi(\Omega_n)\ (\geqslant k)$

- Alice & Bob win the quantum pseudo-telepathy game with $s = k$.

Estimate the gap!!

The independence number of Ω_n

Problem. Estimate $\log_2 \chi(\Omega_n)$ ($\geqslant k$).

Theorem (Galliard ('01), Godsil–Newman ('08))**.**

$$\log_2 \chi(\Omega_n) = k \iff k \in \{1,2,3\} \text{ (i.e., } n \in \{2,4,8\}\text{).}$$

- The independence number $\alpha(\Omega_n)$ of Ω_n is the largest size of an independent set of Ω_n.

Lemma. $\chi(\Omega_n)\,\alpha(\Omega_n) \geqslant |V| = 2^n = 2^{2^k}$.

Proof. A coloring is a partition of V into independent sets. ∎

The main problem of this talk

Lemma. $\chi(\Omega_n)\,\alpha(\Omega_n) \geqslant |V| = 2^n \left(= 2^{2^k}\right)$.

Corollary. $\chi(\Omega_n) \geqslant 2^n/\alpha(\Omega_n)$.

Problem'. Find $\alpha(\Omega_n)$, the independence number of Ω_n.

The main problem of this talk

Problem'. Find $\alpha(\Omega_n)$, the independence number of Ω_n.

- Galliard ('01) found an independent set of Ω_n of size

$$4\sum_{i=0}^{n/4-1}\binom{n-1}{i},$$

and conjectured that this equals $\alpha(\Omega_n)$ for all $n = 2^k$.

- De Klerk & Pasechnik ('07) proved this for $n = 16 = 2^4$, i.e., $\alpha(\Omega_{16}) = 2304$, using the semidefinite programming bound due to Schrijver ('05) based on the Terwilliger algebra.

- This gives $\chi(\Omega_{16}) \geqslant 2^{16}/2304 = 2^{4.83}$.

We need extra $.83$ bit!!

The main result

Theorem (Ihringer–T. ('19)). *For all* $n = 2^k$ $(k \geqslant 2)$*, we have*

$$\alpha(\Omega_n) = 4 \sum_{i=0}^{n/4-1} \binom{n-1}{i}.$$

- The proof is a modification of Frankl's rank argument ('86).
- The proof is just around one page, assuming a bit of knowledge on association schemes.
- I will explain what I think is most interesting in this proof.

A proof sketch

- By Galliard's construction, we know
$$\alpha(\Omega_n) \geqslant 4 \sum_{i=0}^{n/4-1} \binom{n-1}{i}.$$
- Hence it suffices to show that LHS $\leqslant$ RHS.
- Then the proof is reduced to showing the following:

A proof sketch

Claim. *The matrix*

$$\Big(\varphi(\partial(x,y))\Big)_{x,y\in C}$$

is non-singular for any $C \subset \{0,1\}^{n-1}$ *such that*

$$\{\partial(x,y) : x,y \in C\} \subset \{2i : 0 \leqslant i < n/2,\ i \neq n/4\},$$

where

$$\varphi(\xi) = \binom{\xi/2 - 1}{n/4 - 1} = \frac{(\xi/2-1)(\xi/2-2)\cdots(\xi/2-n/4+1)}{(n/4-1)!}.$$

degree $n/4 - 1$

A proof sketch

- Indeed, from every independent set in Ω_n we can construct four such C's, and we have

$$|C| = \operatorname{rank}\left(\varphi(\partial(x,y))\right)_{x,y\in C}$$
$$\leqslant \operatorname{rank}\left(\varphi(\partial(x,y))\right)_{x,y\in\{0,1\}^{n-1}}$$
$$\leqslant \sum_{i=0}^{n/4-1}\binom{n-1}{i},$$

follows from Claim

degree $n/4-1$

where the last $\leqslant$ uses association scheme theory.

A proof sketch

- Recall the matrix

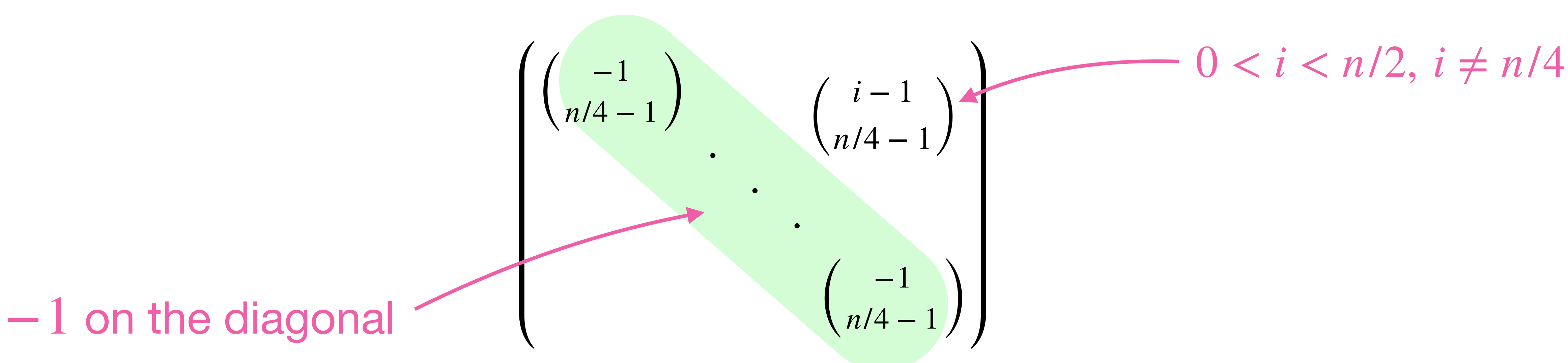
$$\big(\varphi(\partial(x,y))\big)_{x,y\in C},$$

where

$$\{\partial(x,y) : x, y \in C\} \subset \{2i : 0 \leqslant i < n/2,\ i \neq n/4\},$$

and

$$\varphi(\xi) = \binom{\xi/2 - 1}{n/4 - 1} = \frac{(\xi/2-1)(\xi/2-2)\cdots(\xi/2-n/4+1)}{(n/4-1)!}.$$



A proof sketch

- Recall the following result:

Theorem (Lucas). *Let p be a prime, and let*

$$a = \sum_{j=0}^{r} a_j p^j, \quad b = \sum_{j=0}^{r} b_j p^j$$

be p-adic expansions of non-negative integers a and b. Then

$$\binom{a}{b} \equiv \prod_{j=0}^{r} \binom{a_j}{b_j} \pmod{p}.$$

$\binom{\alpha}{\beta} := 0$ if $\alpha < \beta$

$$a = a_r\, a_{r-1} \cdots a_1\, a_{0\,(p)}$$

$$b = b_r\, b_{r-1} \cdots b_1\, b_{0\,(p)}$$

A proof sketch

$$\begin{pmatrix} \binom{-1}{n/4-1} & & \binom{i-1}{n/4-1} \\ & \ddots & \\ & & \binom{-1}{n/4-1} \end{pmatrix}$$

$0 < i < n/2,\ i \neq n/4$

-1 on the diagonal

- As $n/4 - 1 = 2^{k-2} - 1 = \sum_{j=0}^{k-3} 2^j$, we have

$$\binom{i-1}{n/4-1} \equiv 0 \pmod 2 \quad (0 < i < n/2,\ i \neq n/4).$$

$$i - 1 = a_{k-2}\, a_{k-3} \cdots a_1\, a_0\ {}_{(2)}$$
$$n/4 - 1 = \ \ 0 \quad 1 \ \cdots\ 1 \ \ 1 \ \ {}_{(2)}$$

- Hence $\left(\varphi(\partial(x, y))\right)_{x,y \in C} \equiv I \pmod 2$ ← non-singular !! ∎