The 4th workshop on spectral graph theory and related topics
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Complex Fermi surfaces and spectrum of discrete Laplacian on perturbed lattices

Hisashi Morioka
Shibaura Institute of Technology

joint works with
Kazunori Ando (Ehime University) ,
Hiroshi Isozaki (University of Tsukuba)

Contents

I'll talk about eigenvalues embedded in the continuous spectrum of discrete Laplacian on lattices with finite rank perturbations :

- discrete Laplacian on periodic lattices
- Fermi surfaces
- unique continuation properties and lattice structures
- Rellich type uniqueness theorem and its applications

Recent progress on discrete Laplacian

- Spectrum of Laplacian on infinite graphs (Higuchi-Shirai, 2004 / Higuchi-Nomura, 2009)
- Endpoint embedded eigenvalue for higher dim. (Hiroshima-Sakai-Sasaki-Suzuki, 2012)
- Absence of embedded eigenvalues on the square lattice (Isozaki-Morioka, 2014)
- Generalization for some kind of periodic lattices (Ando-Isozaki-Morioka, 2015)
- Endpoint embedded eigenvalue for low dim. (Ogurisu-Higuchi-Nomura, in preparation?)
- Generalization for exponential decaying perturbations (Vesalainen, 2014)
- Generalizations for short-range perturbations (Morioka, in preparation)
- Tree (Colin de Verdiére-Truc, 2013)
- Periodic lattices with pendant vertices (Suzuki, 2013)

Discrete Laplacian with finite rank perturbations

- H. Isozaki, H. Morioka, Inverse Problems and Imaging, **8** (2014), 475-489.
- K. Ando, H. Isozaki, H. Morioka, Ann. Henri Poincaré, online first (2015).

Periodic lattices

e.g.: Square lattice

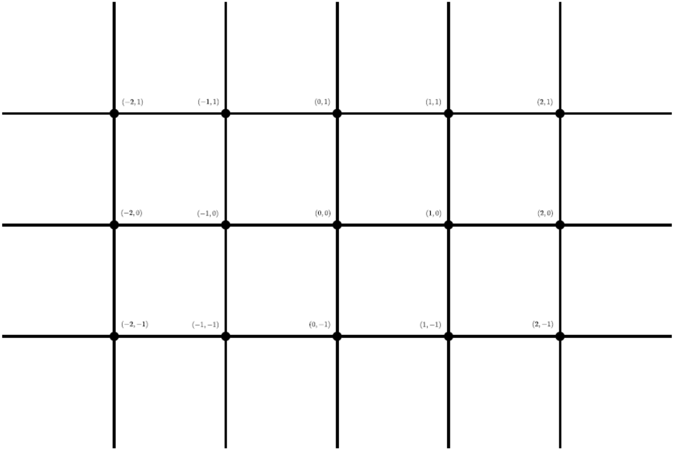

Figure: Square lattice

e.g.: Hexagonal and diamond lattice

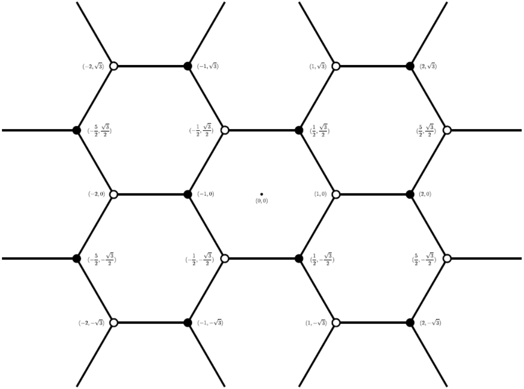

Figure: Hexagonal lattice

e.g.: Kagome lattice

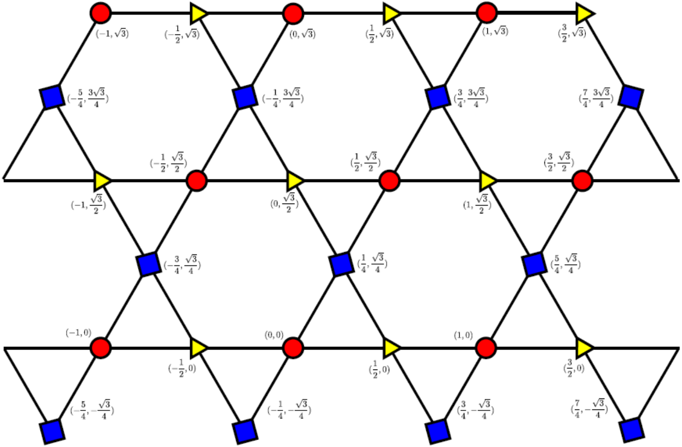

Figure: Kagome lattice

e.g.: Ladder

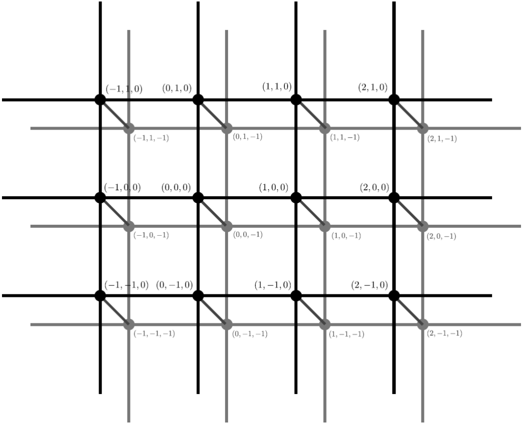

Figure: Ladder of $\mathbf{Z}^2$

e.g.: Graphite

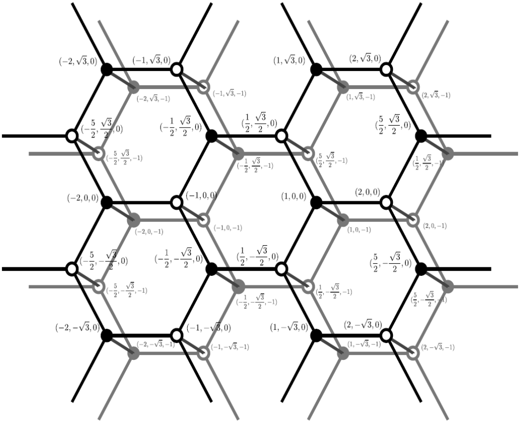

Figure: Graphite

e.g.: subdivision of the 2-D square lattice

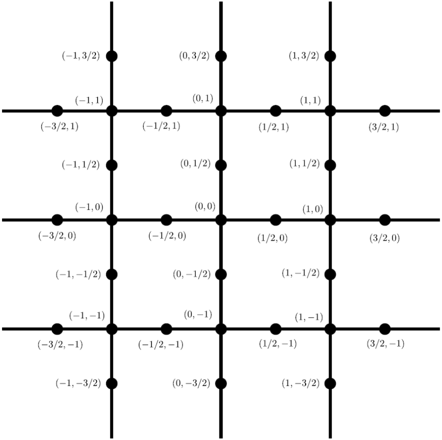

Figure: subdivision

Definition of lattices

These lattice are constructed as a $\mathbf{Z}^d$-covering of a finite graph :

We define $\Gamma_0 = \{\mathcal{L}_0, \mathcal{V}_0, \mathcal{E}_0\}$ by :

For the basis $\mathbf{v}_j$, $j = 1, \cdots, d$ of $\mathbf{R}^d$, we put

$$\mathcal{L}_0 = \{\mathbf{v}(n) \; ; \; n \in \mathbf{Z}^d\}, \quad \mathbf{v}(n) = \sum_{j=1}^{d} n_j \mathbf{v}_j, \quad n \in \mathbf{Z}^d.$$

and, for some points $p_1, \cdots, p_s$ in $\mathbf{R}^d$, we define the set of vertices by

$$\mathcal{V}_0 = \bigcup_{j=1}^{s} (p_j + \mathcal{L}).$$

Moreover, we assume that the set of unoriented edges $\mathcal{E}_0$ is invariant with respect to $\mathbf{Z}^d$-action.

We assume that Γ_0 has no self-loops nor multiple edges.

Discrete Laplacian (transition Laplacian)

For a $\mathbf{C}$-valued function $\widehat{f} = \{\widehat{f}(v)\}_{v\in\mathcal{V}_0}$, we define

$$(\widehat{\Delta}_{\Gamma_0}\widehat{f})(v) = \frac{1}{\deg(v)} \sum_{w\in\mathcal{V}_0,(v,w)\in\mathcal{E}_0} \widehat{f}(w),$$

$$\deg(v) := \sharp\{w \in \mathcal{V}_0 \ ; \ (v,w) \in \mathcal{E}_0\}.$$

By a $\mathbf{R}$-valued scalar potential $\widehat{V}$, we define the discrete Schrödinger equation by

$$(-\widehat{\Delta}_{\Gamma_0} + \widehat{V} - \lambda)\widehat{u} = 0 \quad \text{on} \quad \mathcal{V}_0.$$

Systems of Schrödinger equation

In view of periodic structure of Γ_0, we interpret $\ell^2(\mathcal{V}_0)$ as $\oplus_{j=1}^{s}\ell^2(\mathbf{Z}^d) = \ell^2(\mathbf{Z}^d;\mathbf{C}^s)$. (Other function spaces are also dealt with spaces of $\mathbf{C}^s$-valued functions on $\mathbf{Z}^d$.)
Then we rewrite $\widehat{f} = \{\widehat{f}(v)\}_{v\in\mathcal{V}_0}$ by

$$\widehat{f}(n) = (\widehat{f}_1(n), \cdots, \widehat{f}_s(n)), \quad n \in \mathbf{Z}^d,$$

so that, by the Shift operator $(\widehat{S}_j^{\pm}\widehat{f})(n) = \widehat{f}(n \pm \mathrm{e}_j)$, we have

$$\widehat{H}_0 = -\widehat{\Delta}_{\Gamma_0} = s \times s \text{ symmetric matrix of } \widehat{S}_j^{\pm},$$

$$\widehat{V} = \mathrm{diag}(\widehat{V}_1, \cdots, \widehat{V}_s).$$

$\widehat{H} = \widehat{H}_0 + \widehat{V}$ is bounded and self-adjoint on $\ell^2(\mathbf{Z}^d;\mathbf{C}^s)$.

Fourier transformation

On Γ_0, we can use the Fourier series :

The unitary operator $\mathcal{U}_{\mathcal{L}_0} : \ell^2(\mathbf{Z}^d; \mathbf{C}^s) \to L^2(\mathbf{T}^d; \mathbf{C}^s)$ is defined by

$$(\mathcal{U}_{\mathcal{L}_0} \widehat{f})_j(x) = (2\pi)^{-d/2} \sqrt{\deg_0(j)} \sum_{n \in \mathbf{Z}^d} \widehat{f}_j(n) e^{in\cdot x}, \quad j = 1, \cdots, s,$$

where $\deg_0(j)$ is the degree of vertex in each orbit, and the inner products are

$$(\widehat{f}, \widehat{g})_{\ell^2(\mathbf{Z}^d;\mathbf{C}^s)} = \sum_{j=1}^{s} \sum_{n \in \mathbf{Z}^d} \deg_0(j) \widehat{f}_j(n) \overline{\widehat{g}_j(n)},$$

$$(u, v)_{L^2(\mathbf{T}^d;\mathbf{C}^s)} = \sum_{j=1}^{s} \int_{\mathbf{T}^d} u_j(x) \overline{v_j(x)} dx.$$

Fourier transform of Discrete Laplacian

In the following, we denote $H = H_0 + V$,

$$H_0 = \mathcal{U}_{\mathcal{L}_0} \widehat{H}_0 \mathcal{U}_{\mathcal{L}_0}^* = H_0(x),$$

$$H_0(x) = s \times s \text{ Hermitian matrix with trigonometric-function-entries},$$

$$V = \mathcal{U}_{\mathcal{L}_0} \widehat{V} \mathcal{U}_{\mathcal{L}_0}^*,$$

on the torus $\mathbf{T}^d$.

Then the discrete Schrödinger equation on $\mathbf{T}^d$ is

$$(H_0(x) - \lambda)u + Vu = 0 \quad \text{on} \quad \mathbf{T}^d.$$

Diagonalization of $H_0(x) - \lambda$

Multiplying the co-factor of $H_0(x) - \lambda$, we rewrite $H_0(x) - \lambda$ as

$$H_0(x) - \lambda \to p(x, \lambda) I_s, \quad p(x, \lambda) := \det(H_0(x) - \lambda).$$

Putting eigenvalues of $H_0(x)$ for each $x \in \mathbf{T}^d$ as $\lambda_1(x) \leq \cdots \leq \lambda_s(x)$, we have

$$p(x, \lambda) = \prod_{j=1}^{s} (\lambda_j(x) - \lambda).$$

Essential spectrum of $\widehat{H}_0$

Essential spectrum of $\widehat{H}_0$

The spectrum of $\widehat{H}_0$ is given by

$$\sigma(\widehat{H}_0) = \sigma(H_0) = \bigcup_{x \in \mathbf{T}^d} \text{“eigenvalue of } H_0(x)\text{”}.$$

- In our cases, $\sigma(\widehat{H}_0) = \sigma_{ess}(\widehat{H}_0)$ is a closed interval $\subset [-1, 1]$.
- There may exist $\lambda \in \sigma_p(\widehat{H}_0) \cap \sigma_{ess}(\widehat{H}_0)$! (e.g. Kagome lattice, subdivision lattices)
- Generally, $\sigma(\widehat{H}_0)$ may have some spectral gaps, and eigenvalues with ∞-multiplicities. (c.f. Suzuki, 2013 et al.)

Examples (1)

(1) : Square lattice

- $p(x,\lambda) = -\frac{1}{d}\sum_{j=1}^{d} \cos x_j - \lambda$
- $\sigma(\widehat{H}_0) = [-1, 1]$

(2) : Hexagonal lattice

- $$p(x,\lambda) = \lambda^2 - \frac{3 + 2(\cos x_1 + \cos x_2 + \cos(x_1 - x_2))}{9}$$
- $\sigma(\widehat{H}_0) = [-1, 1]$

(3) : Kagome lattice

- $$p(x,\lambda) = -(\lambda - \tfrac{1}{2})\left(\lambda^2 + \frac{\lambda}{2} - \frac{1 + \cos x_1 + \cos x_2 + \cos(x_1 - x_2)}{8}\right)$$
- $\sigma(\widehat{H}_0) = [-1, 1/2]$

Examples (2)

(4) : Subdivision lattices

- $p(x,\lambda) = (-\lambda)^{d-1}\Big(\lambda^2 - \frac{1}{2d}\big(d + \sum_{j=1}^{d} \cos x_j\big)\Big)$
- $\sigma(\widehat{H}_0) = [-1, 1]$

(5) : (Higher dimensional) ladders

- $p(x,\lambda) = p_+(x,\lambda)p_-(x,\lambda)$
- $$p_\pm(x,\lambda) = \lambda + \frac{1}{2d+1}\Big(2\sum_{j=1}^{d} \cos x_j \pm 1\Big)$$
- $\sigma(\widehat{H}_0) = [-1, 1]$

(6) : Graphite

- $$p(x,\lambda) = \lambda^4 - \frac{\alpha(x)+1}{8}\lambda^2 + \frac{(\alpha(x)-1)^2}{4^4}$$
- $\alpha(x) = 3 + 2(\cos x_1 + \cos x_2 + \cos(x_1 - x_2))$
- $\sigma(\widehat{H}_0) = [-1, 1]$

Remarks

The spectral measure of $\sigma(\widehat{H}_0)$ highly depends on the structure of lattices :

- Geometric structure of complex Fermi surfaces of $\widehat{H}_0$
- Density of states

Fermi surfaces

Real Fermi surfaces, complex Fermi surfaces

The set $M_\lambda \subset \mathbf{T}^d$ is defined by

$$M_\lambda = \{x \in \mathbf{T}^d \ ; \ p(x,\lambda) = 0\}.$$

We extend M_λ to the complex torus $\mathbf{T}^d_{\mathbf{C}} = \mathbf{C}^d/(2\pi\mathbf{Z})^d$, and denote it by

$$M_\lambda^{\mathbf{C}} = \{z \in \mathbf{T}^d_{\mathbf{C}} \ ; \ p(z,\lambda) = 0\}.$$

We split $M_\lambda^{\mathbf{C}}$ into two part, one is the regular part and another is the singular part :

$$M_{\lambda,reg}^{\mathbf{C}} = \{z \in M_\lambda^{\mathbf{C}} \ ; \ \nabla_z p(z,\lambda) \neq 0\},$$
$$M_{\lambda,sng}^{\mathbf{C}} = \{z \in M_\lambda^{\mathbf{C}} \ ; \ \nabla_z p(z,\lambda) = 0\}.$$

Assumption for the Fermi surfaces

Assumption

(A-1) There exists a subset $\mathcal{T}_1 \subset \sigma(\widehat{H}_0)$ such that for $\lambda \in \sigma(\widehat{H}_0) \setminus \mathcal{T}_1$:
(A-1-1) $M^{\mathbf{C}}_{\lambda,sng}$ is a discrete set.
(A-1-2) Each connected component of $M^{\mathbf{C}}_{\lambda,reg}$ intersects with $\mathbf{T}^d$. Each intersection is a $(d-1)$-dimensional real analytic submanifold of $\mathbf{T}^d$.

※ Since $M^{\mathbf{C}}_{\lambda}$ is defined by the trigonometric polynomial $p(z,\lambda)$, we can not define "irreducible factor". However, we can consider an irreducibility in view of the connectivity as complex submanifolds.

Rellich type uniqueness theorem

Rellich type theorem

Theorem (Ando-Isozaki-Morioka, 2015)

We assume (A-1), and let $\lambda \in \sigma(\widehat{H}_0) \setminus \mathcal{T}_1$.
For a function f whose entries are trigonometric polynomials, suppose a distribution u satisfies the equation

$$(H_0(x) - \lambda)u = f \quad \text{on} \quad \mathbf{T}^d,$$

$$\lim_{R\to\infty} \frac{1}{R} \sum_{j=1}^{s} \int_{\mathbf{T}^d} \left|\chi(|\sqrt{-\Delta}| < R)u_j(x)\right|^2 dx = 0.$$

Then entries of u are also trigonometric polynomials.

※ Recalling $\widehat{f} := \mathcal{U}_{\mathcal{L}_0} f$, we have $\sharp \mathrm{supp} \widehat{f} < \infty$. Vesalainen (2014) has generalized our result (Isozaki-Morioka, 2014) for infinite rank perturbations with the condition

$$e^{\gamma\langle n\rangle}\widehat{f} \in \ell^2(\mathbf{Z}^d) \text{ for } \forall\gamma > 0, \ \widehat{f}(n) = 0 \text{ for } \sum_{j=1}^{d-1} |n_j| \leq n_d.$$

Interpretation on the lattice

Corollary

We assume (A-1).
If, for a constant $R_0 > 0$ and $\lambda \in \sigma(\widehat{H}_0) \setminus \mathcal{T}_1$, $\widehat{u}$ satisfies the equation

$$(-\widehat{\Delta}_{\Gamma_0} - \lambda)\widehat{u} = 0 \quad \text{in} \quad |n| > R_0,$$

$$\lim_{R\to\infty} \frac{1}{R} \sum_{j=1}^{s} \sum_{R_0<|n|<R} |\widehat{u}_j(n)|^2 = 0,$$

there exists a constant $R_1 > R_0$ such that $\widehat{u}(n) = 0$ for $|n| > R_1$ i.e. $\sharp\mathrm{supp}\widehat{u} < \infty$.

Application for eigenvalue problems (UCP of Helmholtz type eq.)

Assumption (unique continuation property on lattices)

(A-4) If $\widehat{u}$ satisfies

$$(\widehat{H_0} + \widehat{V} - \lambda)\widehat{u} = 0 \quad \text{on} \quad \mathcal{V}_0$$

and, for a constant $R_1 > 0$, $\widehat{u}(n) = 0$, $|n| > R_1$, then $\widehat{u} = 0$ on whole $\mathcal{V}_0$.

- UCP on lattices is slightly different from elliptic PDE on $\mathbf{R}^d$ or manifolds.
- If we assume (A-1), $\lambda \notin \mathcal{T}_1$ and $\widehat{V} = 0$, UCP holds on our examples.
- If $\widehat{V} \neq 0$, it is not sufficient to assume (A-1) and $\lambda \notin \mathcal{T}_1$. In fact, on kagome lattice and subdivision lattice, UCP does not holds for any $\lambda \in \mathbf{R}$. Moreover, for any $\lambda \in \mathbf{R}$, we can construct $\widehat{V}$ such that $\lambda \in \sigma_p(\widehat{H_0} + \widehat{V})$.

Absence of embedded eigenvalues

Theorem

We assume (A-1) and (A-4).
If $\lambda \in \sigma_{ess}(\widehat{H}) \setminus \mathcal{T}_1$, we have $\lambda \notin \sigma_p(\widehat{H})$ i.e.
$\sigma_p(\widehat{H}) \cap (\sigma_{ess}(\widehat{H}) \setminus \mathcal{T}_1) = \emptyset$.

Sketch of proof.

- Since $\sharp \mathrm{supp} \widehat{V} < \infty$, we can apply the Rellich type theorem for the eigenfunction $\widehat{\psi}_\lambda \in \ell^2(\mathcal{V}_0)$.
- For a sufficiently large $R_1 > 0$, $\widehat{\psi}_\lambda(n) = 0$ for $|n| > R_1$.
- From (A-4), $\widehat{\psi}_\lambda(n) = 0$ on $\mathcal{V}_0$. This is a contradiction.

Examples of $\mathcal{T}_1$ (1)

(1) : Square lattice

- $\sigma(\widehat{H}_0) = [-1, 1]$
- $\mathcal{T}_1 = \{-1, 1\}$

(2) : Hexgonal lattice

- $\sigma(\widehat{H}_0) = [-1, 1]$
- $\mathcal{T}_1 = \{-1, 0, 1\}$

(3) : Kagome lattice

- $\sigma(\widehat{H}_0) = [-1, 1/2]$
- $\mathcal{T}_1 = \{-1, -1/4, 1/2\}$, $1/2 \in \sigma_p(\widehat{H}_0)$.

(4) : subdivision

- $\sigma(\widehat{H}_0) = [-1, 1]$
- $\mathcal{T}_1 = \{-1, 0, 1\}$, $0 \in \sigma_p(\widehat{H}_0)$.

Examples of $\mathcal{T}_1$ (2)

(5) : Ladder

- $\sigma(\widehat{H}_0) = [-1, 1]$
- $\mathcal{T}_1 = \left\{ \frac{2d-1}{2d+1} \leq |\lambda| \leq 1 \right\}$

(6) : Graphite

- $\sigma(\widehat{H}_0) = [-1, 1]$
- $\mathcal{T}_1 = \{1/2 \leq |\lambda| \leq 1\}$

Procedure of UCP

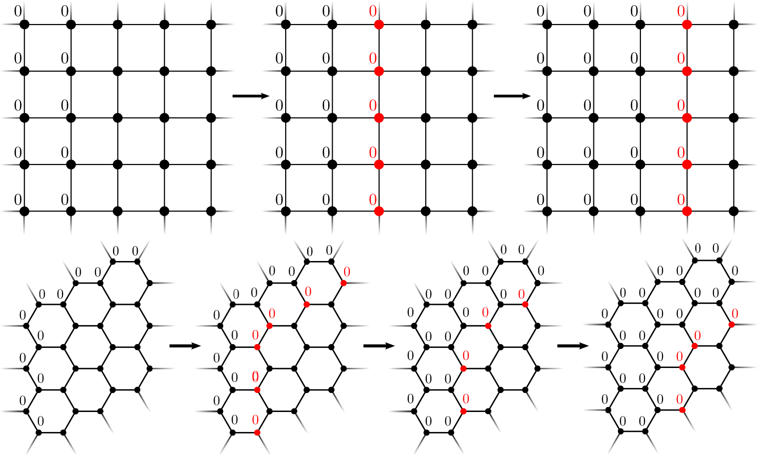

Figure: Unique continuation on lattices

Example of embedded eigenvalue -Kagome lattice-

If we put $\widehat{V}(v) = \alpha$ for $v = x_1, \cdots, x_6$, else $\widehat{V}(v) = 0$, an eigenfunction satisfies $\widehat{u}(v) = (-1)^j$ for $v = x_1, \cdots, x_6$, else $\widehat{u}(v) = 0$ with the eigenvalue $\lambda = \alpha + 1/2$.

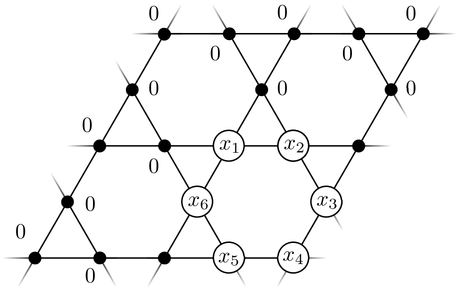

Figure: Eigenfunction with an embedded eigenvalue

Example of embedded eigenvalue -Subdivision lattice-

If we put $\widehat{V}(v) = \alpha$ for $v = x_1, \cdots, x_4$, else $\widehat{V}(v) = 0$, an eigenfunction satisfies $\widehat{u}(v) = (-1)^j$ for $v = x_1, \cdots, x_4$, else $\widehat{u}(v) = 0$with the eigenvalue $\lambda = \alpha$.

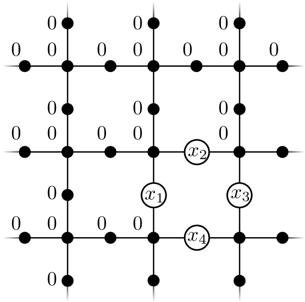

Figure: Eigenfunction with an embedded eigenvalue

Example of embedded eigenvalue -ladder-

For any constant $\alpha \neq 0$, we put $\widehat{V}(0) = \alpha I_2$ and $\widehat{V}(n) = 0$ for $n \neq 0$. For any $\lambda \in \mathcal{T}_1$, we can choose α and construct an eigenfunction which decays at infinity.

Graphite is similar.

Sketch of proof for Rellich type theorem (1)

- Multiplying the co-factor of $H_0(x) - \lambda$, we can diagonalize the equation as
$$p(x, \lambda) I_s u = g,$$
so that we pick up a component. In the following, we deal it with a single equation.
- We obtain $u \in C^\infty(\mathbf{T}^d \setminus M^{\mathbf{C}}_{\lambda,sng})$. In particular, we have $g(x) = 0$ on $M^{\mathbf{C}}_{\lambda,reg} \cap \mathbf{T}^d$.
- In view of (A-1-2), we can extend analytically $g(z) = 0$ to $M^{\mathbf{C}}_{\lambda,reg}$.
- Hence $g(z)/p(z, \lambda)$ is analytic in a neighborhood of $M^{\mathbf{C}}_{\lambda,reg}$.
- From (A-1-1), $M^{\mathbf{C}}_{\lambda,sng}$ is a removable singularity, so that $g(z)/p(z, \lambda)$ is analytic in $\mathbf{T}^d_{\mathbf{C}}$.

Sketch of proof for Rellich type theorem (2)

- Changing the variable $w_j = e^{iz_j}$, we have

$$\frac{g(z)}{p(z,\lambda)} = \frac{G(w)}{P(w,\lambda)} \prod_{j=1}^{d} w_j^{\gamma_j - \beta_j}, \quad G, P \in \mathbf{C}[w_1, \cdots, w_d].$$

- Since LHS is analytic, G/P is also analytic. In particular, $G(w) = 0$ on $\{w \in \mathbf{C}^d \ ; \ P(w,\lambda) = 0\}$.
- Hilbert Nullstellensatz implies that P divides G.
- Therefore $u = g/p$ is a trigonometric polynomial. This implies that $\widehat{u} = \mathcal{U}_{\mathcal{L}_0}^* u$ has a finite support.