Small eigenvalues of the Hodge-Laplacian with sectional curvature bounded below

Junya Takahashi

(Tôhoku University, GSIS)

Riemannian Geometry and Geometric Analysis

21 February 2024

Purpose

The purpose of this talk is to explain some examples of Riemannian metrics on closed manifolds for which the positive eigenvalues of the Hodge-Laplacian tend to 0 (small eigenvalues).

This talk is based on joint works with Colette Anné (Nantes Univ. in France) [AT24a], [AT24b].

[AT24a] C. Anné and J. Takahashi,
Small eigenvalues of the rough and Hodge Laplacians under fixed volume, to appear in Ann. Fac. Sci. Toulouse, (2024). arXiv:2106.12814.

[AT24b] C. Anné and J. Takahashi, (Main Talk)
Small eigenvalues of the Hodge-Laplacian with sectional curvature bounded below, (202b), in preparation.

Contents of this talk

1. Introduction (Notations, Basic facts).
2. Problems on estimates of eigenvalues.
3. Small eigenvalues — Example.
4. Main Theorems A and B.
5. Outline of the proof of Main Theorem A.
6. Outline of the proof of Main Theorem B.
7. Further Studies and Related Topics. (if time permits)

Introduction

Notations

(M^m, g) : m dim. connected, oriented closed Riemann manifold, $(m \geq 2)$

$\Delta = d\delta + \delta d$: Hodge-Laplacian acting on p-forms
(δ is the L^2-adjoint of d)

$\lambda_k^{(p)}(M, g)$: k-th positive eigenvalue of Δ (counted with multiplicity)

The spectrum of Δ consists of non-negative eigenvalues with finite multiplicity:

$$\underbrace{0 = \cdots = 0}_{b_p(M)} < \lambda_1^{(p)}(M, g) \leq \lambda_2^{(p)}(M, g) \leq \cdots \leq \lambda_k^{(p)}(M, g) \leq \cdots \longrightarrow \infty.$$

$b_p(M) = \dim \mathrm{Ker}(\Delta)$: p-th Betti number (Hodge-Kodaira-de Rham)

• The multiplicity of the eigenvalue 0 does not depend on a metric g. (topological invariant)

Basic Properties

(1) Hodge duality: For all $p = 0, 1, \ldots, m$, we have

$$\lambda_k^{(m-p)}(M, g) = \lambda_k^{(p)}(M, g), \qquad \text{(by } \Delta * = * \Delta\text{)}.$$

(2) scaling change of metrics: For a positive constant $a > 0$, we have

$$\lambda_k^{(p)}(M, ag) = a^{-1}\, \lambda_k^{(p)}(M, g) \quad \text{(for any } p, k\text{)}.$$

(3) normalization of the volume: For an m-dim. Riemannian manifold (M, g), we have

$$\overline{g} := \mathrm{vol}(M, g)^{-\frac{2}{m}}\, g \implies \mathrm{vol}(M, \overline{g}) \equiv 1.$$

Therefore, $(2) + (3)$ imply

$$\lambda_k^{(p)}(M, \overline{g}) = \mathrm{vol}(M, g)^{\frac{2}{m}}\, \lambda_k^{(p)}(M, g).$$

Problem on estimates for eigenvalues

Problem 1 (estimates for eigenvalues)

Estimate $\lambda_k^{(p)}(M,g)$ *from above and below, in terms of the geometrical data of* (M,g).

Namely, find *optimal constants* $C_1(M,g,p,k), C_2(M,g,p,k)$ *satisfying that*

$$0 < C_1(M,g,p,k) \leq \lambda_k^{(p)}(M,g) \leq C_2(M,g,p,k).$$

- case $p = 0$ $(p = m)$;

  $C_i(M,g,0,k) = C_i(m, \mathrm{diam}, \inf \mathrm{Ric}, k)$. $i = 1, 2$.

  diam = diameter, $\inf \mathrm{Ric}$ = infimum of the Ricci curvature.

  (by Cheeger, Cheng, Gallot, Li-Yau, Gromov, et al. $\cdots$ well-known)

- cases $1 \leq p \leq m-1$;

  still unknown (important problem) ← very difficult

Small eigenvalues

In the case of $1 \leq p \leq m-1$, the estimate for $p=0$ does not hold in general.

Example 2 (Small eigenvalues)

$\exists \{(M^m, g_i)\}_{i=1}^{\infty}$: *a sequence of m-dim. closed Riem. mfds with*

$$\mathrm{diam}(M, g_i) \leq D, \quad \mathrm{Ric}_{(M,g_i)} \geq -(m-1)K^2$$

(D, K are indep. of i) such that

$$\lambda_k^{(p)}(M, g_i) \longrightarrow 0 \quad (i \to \infty).$$

These are called small eigenvalues.

- These examples are given by collapsing of Riemannian manifolds.
- In our study, under a fixed manifold M, we consider metrics g_i.

Hopf $\mathbb{S}^1$-bundle (Colbois-Courtois [CC90])

$\pi : (\mathbb{S}^{2n+1}, g) \xrightarrow{\mathbb{S}^1} (\mathbb{CP}^n, h)$: the Hopf $\mathbb{S}^1$-bundle
(considered as the Riemannian submersion).

For $\varepsilon > 0$, the collapsing metrics g_ε on $\mathbb{S}^{2n+1}$ are defined as

$$g_\varepsilon := g_H \oplus \varepsilon^2 \ g_V ,$$

where g_H and g_V are the horizontal and the vertical parts of g, resp.

Then,

$$\operatorname{diam}(\mathbb{S}^{2n+1}, g_\varepsilon) \leq D, \quad |K_{(\mathbb{S}^{2n+1}, g_\varepsilon)}| \leq K.$$

As $\varepsilon \to 0$, we have for $q = 0, 1, \ldots n$,

$$\lambda_1^{(2q)}(\mathbb{S}^{2n+1}, g_\varepsilon) \longrightarrow 0 \ (= \lambda_0^{(2q)}(\mathbb{CP}^n, h)),$$

$$\pi^*(\omega^q) \longrightarrow \ \omega^q$$

where $\omega^q = \underbrace{\omega \wedge \cdots \wedge \omega}_{q \text{ times}}$ for the Kähler form ω on $(\mathbb{CP}^n, h)$.

The case of odd degrees follows form the Hodge duality:
$\lambda_1^{(2q+1)} = \lambda_1^{(2(n-q))}$. □

Problems on Small eigenvalues

Problem 3 (Small eigenvalues)

When does there exist small eigenvalues ? How many small eigenvalues ?

Then, in what situations are Riemannian manifolds ?

For now, we want to construct many kinds of examples of families of closed Riemannian manifolds with small eigenvalues.

Small eigenvalues on m-sphere $\mathbb{S}^m$

Theorem 4 (Anné and T. (2024) [AT24a])

Fix the dim. $m \geq 2$ *and the degree* p *with* $1 \leq p \leq m-1$.

$\exists \ \{\overline{g}_{p,L}\}_{L \geq 1}$: 1-*parameter family of Riem. metrics on* m-*sphere* $\mathbb{S}^m$ *with*

$$\mathrm{vol}(\mathbb{S}^m, \overline{g}_{p,L}) \equiv 1, \quad K_{(\mathbb{S}^m, \overline{g}_{p,L})} \geq 0$$

s.t. for all $k = 1, 2, \ldots,$

$$\lambda_k^{(p)}(\mathbb{S}^m, \overline{g}_{p,L}) \longrightarrow 0 \qquad (L \longrightarrow \infty).$$

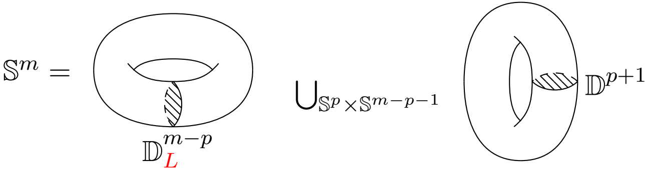
[Remark]

- the diameter $\mathrm{diam}(M, \overline{g}_{p,L}) \longrightarrow \infty$ $(L \longrightarrow \infty)$.

Construction of the metrics in Thm. 4 (outline)

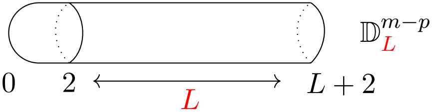
(1) decompose $\mathbb{S}^m$ like a p-dim. surgery:

$$\mathbb{S}^m = \left(\mathbb{S}^p \times \mathbb{D}^{m-p}_L\right) \cup_{\mathbb{S}^p \times \mathbb{S}^{m-p-1}} \left(\mathbb{D}^{p+1} \times \mathbb{S}^{m-p-1}\right).$$



(2) take Riemannian metrics $g_{p,L}$ on $\mathbb{D}^{m-p}_L$ such that $\mathbb{D}^{m-p}_L$ looks like a long cylinder, as $L \longrightarrow \infty$ (see the figure below).

The sectional curvature is non-negative: $K \geq 0$.



(3) normalize the volume of $\mathbb{S}^m$: $\overline{g}_{p,L} := \mathrm{vol}(\mathbb{S}^m, g)^{-\frac{2}{m}}\, g_{p,L}$.

Note that the sectional curvature is still non-negative. □

We also give lower bounds.

Theorem 5 (large eigenvalues)

In particular, for $q \neq 1, p, p+1, m-p-1, m-p, m-1, m$, *we have*

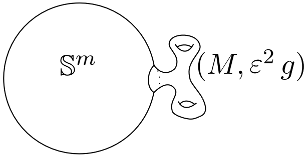
$$\lambda_1^{(q)}(\mathbb{S}^m, \overline{g}_{p,L}) \geq C\big(AL + B\big)^{\frac{2}{m}} \longrightarrow \infty \quad (L \longrightarrow \infty),$$

where $A, B, C > 0$ *are some constants indep. of* L.

The case of general manifolds

We glue this sphere obtained by Theorem 4 to a general manifold M, by using a gluing theorem for the eigenvalues [AT12].

But, the sectional curvature on the gluing part diverge to $\pm\infty$



Theorem 6 (Anné and T. [AT24a])

$M^m : m \geq 2$ *dim. conn. ori. closed manifold.*

p : *fix a degree of forms with* $1 \leq p \leq m-1$.

For any $\varepsilon > 0$ *and any* $k \geq 1$, *there exists a Riem. metric* $\overline{g}_{p,\varepsilon}$ *on* M *with volume* 1 *such that*

$$\lambda_k^{(p)}(M, \overline{g}_{p,\varepsilon}) < \varepsilon \,.$$

Next Problem

Problem 7

For a general manifold M, can we impose any curvature constraints ?

In particular, can we obtain the same results in the case of $K_M \geq -K^2$?

[Remark]

- M has a topological obstruction to have a non-negative curvature.

(e.g. Bochner thm.: $\mathrm{Ric} \geq 0 \implies b_1(M) \leq b_1(\mathbb{T}^m) = m$)

Therefore, we consider the case of sectional curvature $K_M \geq -K^2$.
This case has no topological obstruction.

[Answer] ⋯Yes (Main Theorem))

Main Theorem A

Theorem 8 (Main Theorem A (Anné and T. [2024b]))

M^m : *conn. oriented, closed* m*-manifold* $(m \geq 2)$
p $(0 \leq p \leq m)$: *the degree of forms.*
$k \geq 1$: *the number of the eigenvalues.*

For any $\varepsilon > 0$, *there exist Riem. metrics* $\overline{g}_{\varepsilon,p,k}$ *on* M *with*

$$\mathrm{vol}(M, \overline{g}_{\varepsilon,p,k}) \equiv 1, \quad K_{(M,\overline{g}_{\varepsilon,p,k})} \geq -K^2$$

such that

$$\lambda_k^{(p)}(M, \overline{g}_{\varepsilon,p,k}) \longrightarrow 0 \qquad (\varepsilon \longrightarrow 0).$$

[Remarks]

- $\overline{g}_{\varepsilon,k,p}$ depends also on the degree p and the number k.
- $\mathrm{diam}(M, \overline{g}_{\varepsilon,p,k}) \longrightarrow \infty$ $(\varepsilon \longrightarrow 0)$.

Main Theorem B (uniformness for the degree)

We can choose Riem. metrics which do not depend on the degree p of forms.

Theorem 9 (Main Theorem B (Anné and T. [2024b]))

M^m : *m-dim. conn. oriented closed manifold* $(m \geq 2)$
$k \geq 1$: *the number of the eigenvalues.*

For any $\varepsilon > 0$, there exist Riem. metrics $\overline{g}_{\varepsilon,k}$ on M with

$$\mathrm{vol}(M, \overline{g}_{\varepsilon,k}) \equiv 1, \qquad K_{(M,\overline{g}_{\varepsilon,k})} \geq -K^2$$

such that for all $p = 0, 1, \ldots, m$

$$\lambda_k^{(p)}(M, \overline{g}_{\varepsilon,k}) \longrightarrow 0 \qquad (\varepsilon \longrightarrow 0).$$

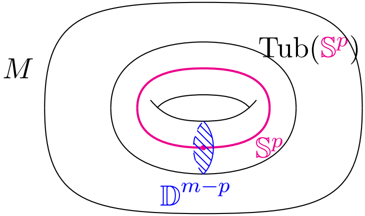
Outline of the proof of Main Thm. A

- We may assume $0 \le p \le m-2$ (by Hodge duality).

(1) Take an embedding $\mathbb{S}^p \hookrightarrow M$ whose normal bundle is trivial:

$$\mathrm{Tub}(\mathbb{S}^p) \cong \mathbb{S}^p \times \mathbb{D}^{m-p}.$$

$g_{p,M}$: any Riemannian metric on M which is product on $\mathrm{Tub}(\mathbb{S}^p)$.

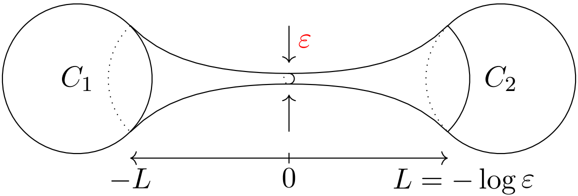


(2) We decompose M into two components: $M = H_1 \cup H_2$.

$$M = H_1 \cup H_2 = \left(\mathbb{S}^p \times \mathbb{D}^{m-p}\right) \cup \left(M \setminus (\mathbb{S}^p \times \mathbb{D}^{m-p})\right).$$

$(\mathbf{3})$ We glue the connected sums of the k copies of C_ε in series to the disk $\mathbb{D}^{m-p}$ as follows:

$(\mathbf{3-1})$ The hyperbolic dumbbell C_ε (Boulanger-Courtois [BC22])



The central part of C_ε is the hyperbolic cylinder, whose Riemannian metric g_ε is expressed as

$$g_\varepsilon := dr^2 \oplus \varepsilon^2 \cosh^2(r) g_{\mathbb{S}^{m-p-1}} \ (-L \leq r \leq L = -\log\varepsilon).$$

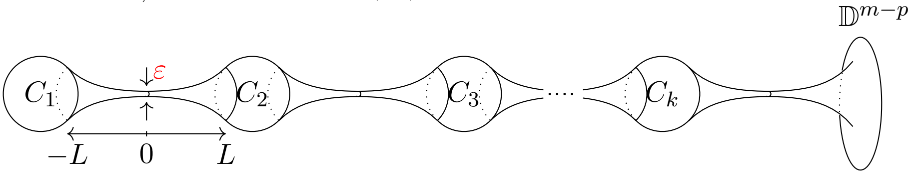
The metric g_ε on the two-sides bumps $C_1, C_2 (\cong \mathbb{S}^{m-p})$ does not depend on ε.

Properties $(\star)$

1. the sectional curvature : $K_{g_\varepsilon} \geq -1$.
2. the volume : $0 < V_1 \leq \mathrm{vol}(C_\varepsilon, g_\varepsilon) \leq V_2$ for $V_1, V_2 > 0$ indep. of ε.

$(\mathbf{3-2})$ the connected sums of the k copies of the hyperbolic dumbbells in series: $C_{k,\varepsilon} = \overset{k}{\sharp} C_\varepsilon$

$C_{k,\varepsilon} = \overset{k}{\sharp} C_\varepsilon$ with the metric $g_{C_{k,\varepsilon}}$: the connected sums of the k copies of the hyperbolic dumbbells C_ε in series (below).
We glue $C_{k,\varepsilon}$ to $\mathbb{D}^{m-p}$ of $\mathrm{Tub}(\mathbb{S}^p)$.



$C_{k,\varepsilon}$ also satisfies the Properties $(\star)$.

$\textbf{(4)}$ We define Riem. metrics $g_{\varepsilon,p,k}$ on M as:

$$g_{\varepsilon,p,k} := \begin{cases} g_{\mathbb{S}^p} \oplus g_{C_{k,\varepsilon}} & \text{on } H_1 = \mathrm{Tub}(\mathbb{S}^p) = \mathbb{S}^p \times \mathbb{D}^{m-p}, \\ g_{p,M} & \text{on } H_2 = \overline{M \setminus H_1}. \end{cases}$$

$\textbf{(5)}$ Finally, normalize the volume to be 1:

$$\overline{g}_{\varepsilon,p,k} := \mathrm{vol}(M, g_{\varepsilon,p,k})^{-\frac{2}{m}} g_{\varepsilon,p,k} \text{ on } M.$$

Since $0 < V_1 \leq \mathrm{vol}(C_\varepsilon, g_\varepsilon) \leq V_2$, the following holds:

1. $K_{\overline{g}_{\varepsilon,p,k}} \geq -K^2$.
2. $\mathrm{vol}(M, \overline{g}_{\varepsilon,p,k}) \equiv 1$.

□

Small eigenvalues

Lemma 10 (Small eigenvalues)

For all $p = 0, 1, 2, \ldots, m-2$ *and* $k \geq 1$*, we have*

$$\lambda_k^{(p)}(M, \overline{g}_{\varepsilon,p,k}) \longrightarrow 0 \quad (\varepsilon \longrightarrow 0)$$

[Note]

• In the case of $p = m-1, m$, by the Hodge duality, we can deduce to the case of $p = 1, 0$.

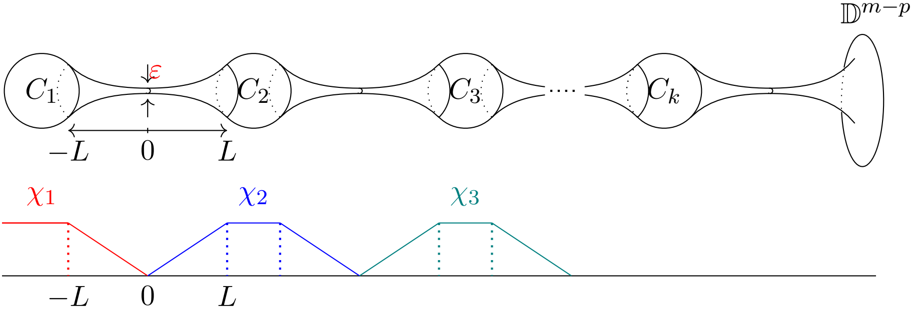
$\Longrightarrow$ Lemma 10，Main Theorem A are ture for $p = m-1, m$.

Outline of the proof of Lemma 10

- To use the min-max principle, we construct k test p-forms.

$\chi_i(r)$: the linear cut-off function as in the following figure.

$\mathrm{supp}(\chi_i) \cap \mathrm{supp}(\chi_j) = \emptyset$ $(i \neq j)$, disjoint.



$$\chi_1 := \begin{cases} 1 & \text{on } C_1, \\ -\dfrac{r}{L} & \text{for } -L \leq r \leq 0, \\ 0 & \text{for } 0 \leq r. \end{cases}$$

Then, we take the test p-forms φ_i as follows:

$$\varphi_i := \begin{cases} \chi_i(r)\, v_{\mathbb{S}^p} & \text{on } H_1 = \mathbb{S}^p \times \mathbb{D}^{m-p}, \\ 0 & \text{on } H_2 = \overline{M \setminus H_1}, \end{cases}$$

where $v_{\mathbb{S}^p}$ is the volume form of $\mathbb{S}^p$.

Then, φ_i is co-closed form, from the min-max principle, we obtain our desired estimate:

$$\begin{aligned} \lambda_k^{(p)}(M, \overline{g}_{\varepsilon,p,k}) &\leq \max_{i=1,\dots,k+b_p(M)} \frac{\| d\,\varphi_i \|^2_{L^2(M,\overline{g}_{\varepsilon,p,k})}}{\| \varphi_i \|^2_{L^2(M,\overline{g}_{\varepsilon,p,k})}} \\ &\leq \cdots\cdots \\ &\leq \frac{C}{|\log \varepsilon|} \longrightarrow 0 \quad (\varepsilon \longrightarrow 0). \end{aligned}$$

□

Outline of the proof of Main Thm. B

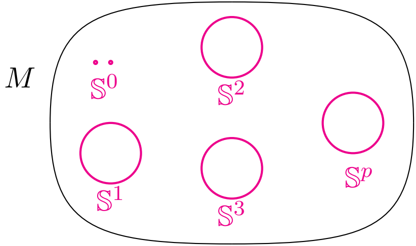
- uniformness for the degree p

Riemannian metrics $\overline{g}_{\varepsilon,p,k}$ in Main Thm. A depend on the degree p.

We take disjoint embedded spheres $\mathbb{S}^0, \mathbb{S}^1, \dots, \mathbb{S}^{m-2}$, and apply the way of the construction in Main Thm A to each tubular neighborhood.

$\Longrightarrow$ Riemannian metrics $\overline{g}_{\varepsilon,k}$ on M do NOT depend on all the degree $p = 0, 1, 2, \dots, m$.

We obtain Main Thm. B. □



Further Studies and Related Topics

Remarks

In Main Theorems A, B, the diameters diverge:

$$\mathrm{diam}(M, \overline{g}_{\varepsilon,p,k}) \longrightarrow \infty \quad (\varepsilon \longrightarrow 0).$$

Problem 11

How about the case of $\mathrm{diam}(M, g) < \infty$ *in addition ?*

This is a non-collapsing case.

- A non-collapsing case means that the dimension of the limit space is not decreasing (is unchanged).

In this case, J. Lott posed the following conjecture in 2004, [Lo04].

Lott conjecture

Conjecture 12 (Lott conjecture (2004, [Lo04]))

(M^m, g) : m-dim. conn. oriented closed Riem. manifold,

If (M, g) satisfies

$$K_{(M,g)} \geq -K, \quad \mathrm{diam}(M,g) \leq D, \quad \mathrm{vol}(M,g) \geq v > 0 \qquad (\sharp)$$

for some constants $K, D, v > 0$ (non-collapsing case), then there would exist a uniform constant $C = C(m, K, D, v) > 0$ such that

$$\lambda_1^{(p)}(M, g) \geq C(m, K, D, v) > 0.$$

(In particular, C would be independent of the degree p.).

Remarks and Comments

• It would be considered that this conjecture holds true.

• If the Lipschitz stability theorem stated by G. Perelman (unpublished) would hold true, then the Lott conjecture also holds true.

The Lipschitz stability theorem states that:

$$(M_1^m, g_1) \underset{d_{GH}}{\sim} (M_2^m, g_2) \ \text{ with } (\sharp) \Longrightarrow (M_1, g_1) \underset{\text{bi-Lipschitz}}{\cong} (M_2, g_2).$$

(This theorem is a statement for Alexandrov spaces.)
The Lipschitzness ensures a control of the norm of all the 1st derivatives.

• The case of $\mathrm{Ric}_{(M,g)} \geq -K$, instead of $K_{(M,g)} \geq -K$.

$\cdots$ It would be considered that the same statement does NOT hold.
(The estimate for the Betti numbers by Gromov does not hold.)

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