L^2-harmonic forms on incomplete Riemannian manifolds with positive Ricci curvature

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Dedicated to the memory of Professor Ahmad El Soufi

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Abstract

We construct an incomplete Riemannian manifold with positive Ricci curvature which has non-trivial L^2-harmonic forms and on which the L^2-Stokes theorem does not hold. Therefore, a Bochner-type vanishing theorem does not hold for incomplete Riemannian manifolds.

1 Introduction

The Stokes theorem or the Green formula plays a very important role in geometry and analysis on manifolds. For example, we recall the proof of the Bochner vanishing theorem (e.g. [Jo11] p.185, Theorem 4.5.2).

Theorem 1.1 (Bochner vanishing theorem)**.** *Let* (M, g) *be a connected oriented closed Riemannian manifold. If the Ricci curvature* $\mathrm{Ric} > 0$ *on* M*, then the first cohomology group* $H^1(M; \mathbb{R}) = 0$.

From the proof of the Bochner vanishing theorem, it follows that if the Stokes theorem does not hold on an incomplete Riemannian manifold of positive Ricci curvature, then the Bochner vanishing theorem for it might not hold. It is a natural question to ask whether or not the Stokes theorem on general incomplete Riemannian manifolds hold. Indeed, Cheeger in [Ch80] studied the Stokes theorem and the Hodge theory on Riemannian manifolds with conical singularities, more generally, Riemannian pseudomanifolds. The analysis on pseudomanifolds is, by definition,

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the L^2-analysis on the regular set which excludes the singular points. Then, there are many valuable results on Riemannian pseudomainfolds (e.g. [Ch83], [CGM82]). Indeed, Cheeger, Goresky and MacPherson [CGM82] stated that the L^2-cohomology groups of the regular sets of Riemannian pseudomanifolds are isomorphic to the intersection cohomology groups with the lower middle perversities. These studies have still been developing by many mathematicians (see [KW06], [Na83], [Yo94], [Bei14b]). Recently, Albin, Leichtnam, Mazzeo and Piazza [ALMP18] studied the Hodge theory on more general singular spaces, which were called Cheeger spaces.

On the other hand, Cheeger ([Ch80] p. 140, Theorem 7.1 and [Ch86] p. 34, Theorem 3) proved that generalized Bochner-type vanishing theorems hold on some Riemannian pseudomanifolds with a kind of "positive curvature". This kind of "positive curvature" seems to behave like a positive curvature operator.

But it seems that there are no concrete examples where a Bochner-type vanishing theorem does not hold. So we construct a simple concrete example where a Bochner-type vanishing theorem does not hold. Note that a Bochner-type vanishing theorem holds for complete Riemannian manifolds [Do81].

In the present paper, we give an incomplete Riemannian manifold with positive Ricci curvature for which a Bochner-type vanishing theorem does not hold. The construction of our manifold is the following way. Let (N^n, h) be a connected oriented closed Riemannian manifold of dimension n. We consider the suspension $\Sigma(N)$ of N, and equip the smooth set of $\Sigma(N)$ with a Riemannian metric g. We denote by $\overline{M}$ the suspension of N:

$$\overline{M} := \Sigma(N) = [0,\pi] \times N \big/ \sim,$$

where the equivalent relation is

$$(r_1, y_1) \sim (r_2, y_2) \overset{\text{equiv.}}{\iff} r_1 = r_2 = 0 \text{ or } \pi$$

for $(r_1, y_1), (r_2, y_2) \in [0,\pi] \times N$. Let $M = \overline{M}_{\mathrm{reg}}$ be the regular set of $\overline{M}$, which consists of all smooth points of $\overline{M}$, i.e., $\overline{M}_{\mathrm{reg}} = (0,\pi) \times N$. The singular set is $\overline{M}_{\mathrm{sing}} := \overline{M} \setminus \overline{M}_{\mathrm{reg}}$, i.e., two vertices corresponding to $r = 0, \pi$. We define an incomplete Riemannian metric g on this smooth part $M = (0,\pi) \times N$ as

$$g := dr^2 \oplus \sin^{2a}(r) h$$

for some constant $0 < a < 1$. In fact, we take $a = \frac{1}{n}$. This metric is a warped product metric with the warping function $\sin^a(r)$. Then, our main theorem is stated as follows:

Theorem 1.2. *There exists an incomplete Riemannian manifold (M^m, g) of dimension $m \geq 2$ satisfying the following four properties:*

(1) *the Ricci curvature of* (M, g) *is* $\mathrm{Ric} \geq K > 0$ *for some constant* $K > 0$*;*

(2) *there exist non-trivial* L^2*-harmonic* p*-forms on* (M, g) *for all* $1 \leq p \leq m-2$*;*

(3) *the* L^2*-Stokes theorem for all* $1 \leq p \leq m-2$ *does not hold on* (M, g)*;*

(4) *the capacity of the singular set satisfies* $\mathrm{Cap}(\overline{M}_{\mathrm{sing}}) = 0$.

Remark 1.3. (*i*) *In the case of* $p = 1$*, Theorem 1.2 implies that a Bochner-type vanishing theorem does not hold for an incomplete Riemannian manifold with* $\mathrm{Ric} \geq K > 0$.

(*ii*) *The curvature operator on* (M, g) *is not positive. But we do not know whether or not the Weitzenböck curvature tensor* F_p *is positive, where* F_p *is the curvature term in the Weitzenböck formula for* p*-form* φ*:*

$$-\frac{1}{2}\Delta(|\varphi|_g^2) = -\langle \Delta\varphi, \varphi\rangle_g + |\nabla\varphi|_g^2 + \langle F_p\varphi, \varphi\rangle_g. \tag{1.1}$$

Therefore, we do not apply the Bochner-type vanishing theorem for all p*-forms by Gallot and Meyer [GM75], p.262, Proposition* 0.9*. Note that the Weitzenböck curvature tensor is estimated below by a lower bound of the curvature operator (e.g. [Pe16], p. 346, Corollary* 9.3.4*).*

(*iii*) *For harmonic* 1*-form* $\varphi = d\theta$ *on* $\mathbb{T}^n$*, by the equation* (1.1) *and* $F_1 = \mathrm{Ric}$*, there exists non-constant subharmonic function* $|d\theta|_g^2 = \sin^{-2/n}(r)$ *on* $M = (0, \pi) \times \mathbb{T}^n$*, that is,* $\Delta(|\varphi|_g^2) \leq 0$ *on* M*.*

The present paper is organized as follows: In Section 2, we recall two important closed extensions of the exterior derivative d, which are $d_{\max}$ and $d_{\min}$, and the L^2-Stokes theorem on Riemannian manifolds with conical singularity by Cheeger [Ch80]. In Section 3, we calculate L^2-harmonic forms on a warped product Riemannian manifold and the capacity of the vertex. In Section 4, the final section, we prove Theorem 1.2.

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2 L^2-Stokes theorem.

Let (M^m, g) be a connected oriented (possibly incomplete) Riemannian manifold of dimension m. We denote by $\Omega_0^p(M)$ the set of all smooth p-forms on M with compact support, and by d_p the exterior derivative acting on smooth p-forms. We consider the de Rham complex $d_p : \Omega_0^p(M) \longrightarrow \Omega_0^{p+1}(M)$ for $p = 0, 1, 2, \ldots, m-1$ with $d_{p+1} \circ d_p \equiv 0$. By using the Riemannian metric g, we define the L^2-inner product on $\Omega_0^p(M)$ as

$$(\varphi, \psi)_{L^2(\Lambda^p M, g)} := \int_M \langle \varphi, \psi \rangle_g \, d\mu_g$$

for any $\varphi, \psi \in \Omega_0^p(M)$, where $d\mu_g$ is the Riemannian measure and $\langle \ , \ \rangle_g$ is the fiber metric on the exterior bundle $\Lambda^p T^* M$ induced from the Riemannian metric g. The space of L^2 p-forms $L^2(\Lambda^p M, g)$ is the completion of $\Omega_0^p(M)$ with respect to this L^2-norm.

Next, we consider the completion of the exterior derivative d_p, which induces a Hilbert complex introduced by Brüning and Lesch [BL92], p.90. There are two important closed extensions of d_p, one of which is the maximal extension $d_{p,\max}$ and the other is the minimal extension $d_{p,\min}$.

Definition 2.1 (maximal extension $d_{p,\max}$). *The maximal extension $d_{p,\max}$ is the operator acting on the domain:*

$$\mathrm{Dom}(d_{p,\max}) := \Big\{ \varphi \in L^2(\Lambda^p M, g) \,\Big|\, \text{There exists } \psi \in L^2(\Lambda^{p+1} M, g) \text{ such that } (\varphi, \delta_{p+1}\eta)_{L^2(\Lambda^p M, g)} = (\psi, \eta)_{L^2(\Lambda^{p+1} M, g)} \text{ for any } \eta \in \Omega_0^{p+1}(M) \Big\},$$

and in this case, we write

$$d_{p,\max}\varphi = \psi.$$

In other words, $\mathrm{Dom}(d_{p,\max})$ *is the largest set of differential p-forms $\varphi \in L^2(\Lambda^p M, g)$ such that the distributional derivative $d_p\varphi$ is also in $L^2(\Lambda^{p+1} M, g)$.*

Definition 2.2 (minimal extension $d_{p,\min}$). *The minimal extension $d_{p,\min}$ is given by the closure with respect to the graph norm of d_p in $L^2(\Lambda^p M, g)$, that is,*

$$\mathrm{Dom}(d_{p,\min}) := \Big\{ \varphi \in L^2(\Lambda^p M, g) \,\Big|\, \text{There exists } \{\varphi_i\}_i \in \Omega_0^p(M) \text{ such that } \varphi_i \to \varphi,\ d_p\varphi_i \to \psi \in L^2(\Lambda^{p+1} M, g)\ (L^2\text{-strongly}) \Big\}.$$

and in this case, we write

$$d_{p,\min}\varphi = \psi.$$

In other words, $d_{p,\min}$ is the smallest closed extension of d_p, that is, $d_{p,\min} = \overline{d_p}$.

It is obvious that

$$\Omega_0^p(M) \subset \mathrm{Dom}(d_{p,\min}) \subset \mathrm{Dom}(d_{p,\max}).$$

In the same manner, from the co-differential operator $\delta_p := (-1)^{mp+m+1} * d_{m-p} * : \Omega_0^p(M) \longrightarrow \Omega_0^{p-1}(M)$, where $*$ is the Hodge $*$-operator on (M,g), we can define the maximal extension $\delta_{p,\max}$ and the minimal extension $\delta_{p,\min}$. These operators are mutually adjoint, that is,

$$(\delta_{p+1,\min})^* = d_{p,\max}, \quad (\delta_{p+1,\max})^* = d_{p,\min}. \tag{2.1}$$

Note that min and max are exchanged.

Now, we recall the definition of the L^2-Stokes theorem for p-forms (see Cheeger [Ch80] p.95 (1,7), [GL02] p.72, Definition 2.2, [Beh09] p.40, Definition 4.1).

Definition 2.3 (L^2-Stokes theorem)**.** *Let (M^m, g) be a connected oriented Riemannian manifold. The L^2-Stokes theorem for p-forms holds on (M,g), if*

$$(d_{p,\max}\varphi, \psi)_{L^2(\Lambda^{p+1} M,g)} = (\varphi, \delta_{p+1,\max}\psi)_{L^2(\Lambda^p M,g)} \tag{2.2}$$

for any $\varphi \in \mathrm{Dom}(d_{p,\max})$ and $\psi \in \mathrm{Dom}(\delta_{p+1,\max})$.

For complete Riemannian manifolds, the L^2-Stokes theorem for all p-forms always holds (Gaffney [Ga51], [Ga54]).

Since the equation (2.2) implies $d_{p,\max} = (\delta_{p+1,\max})^*$, the L^2-Stokes theorem for p-forms holds if and only if $d_{p,\min} = d_{p,\max}$, i.e., a closed extension of d_p is unique.

Now, for any $\varphi \in \mathrm{Dom}(d_{p,\max})$ and $\psi \in \mathrm{Dom}(\delta_{p+1,\max})$, we see that

$$\begin{aligned}(d_{\max}\varphi,\ \psi)_{L^2(\Lambda^{p+1} M,g)} - (\varphi,\ \delta_{\max}\psi)_{L^2(\Lambda^p M,g)} &= \int_M \langle d_{\max}\varphi,\ \psi\rangle d\mu_g - \int_M \langle \varphi,\ \delta_{\max}\psi\rangle d\mu_g \\ &= \int_M d_{L^1,\max}(\varphi \wedge *_g\psi),\end{aligned}$$

where the last $d_{L^1,\max}$ is the maximal extension of d_{m-1} between $L^1(\Lambda^* M, g)$, that is, the domain is $\{\, \omega \in L^1(\Lambda^{m-1} M, g) \,|\, d\omega \in L^1(\Lambda^m M, g)$ (in the distribution sense) $\}$. Therefore, we have

Lemma 2.4. *The L^2-Stokes theorem for p-forms holds on (M,g) if and only if*

$$\int_M d_{L^1,\max}(\varphi \wedge *_g\psi) = 0$$

for any $\varphi \in \mathrm{Dom}(d_{p,\max})$ and $\psi \in \mathrm{Dom}(\delta_{p+1,\max})$.

Remark 2.5. *Gaffney ([Ga54] p.141, Theorem) proved the L^1-Stokes theorem, or the special Stokes theorem, for oriented complete Riemannian manifolds: If any smooth $(m-1)$-form ω on an oriented complete Riemannian manifold of dimension m such that $\omega, d\omega$ are in $L^1(\Lambda^* M, g)$, then*

$$\int_M d\omega = 0.$$

This L^1-Stokes theorem implies the L^2-Stokes theorem for all p-forms, but the inverse does not hold (see Grigor'yan and Masamune [GM13] p.614, Proposition 2.4).

We recall connected oriented compact Riemannian manifolds with conical or horn singularity (Cheeger [Ch80], [Ch83]). Let (N^n, h) be a connected oriented closed Riemannian manifold of dimension n, and let M_1^m be a connected oriented compact manifold of dimension $m = n+1$ with the boundary $\partial M_1 = N$. Let $f : I = [0, l] \longrightarrow \mathbb{R}_+$ be a smooth function with $f(0) = 0$ and $f(r) > 0$ for $r > 0$. The metric f-horn $C_f(N)$ over (N, h) is defined as the metric space

$$C_f(N) = I \times N \Big/ \sim,$$

where the equivalent relation is

$$(r_1, y_1) \sim (r_2, y_2) \overset{\text{equiv.}}{\Longleftrightarrow} r_1 = r_2 = 0$$

for $(r_1, y_1), (r_2, y_2) \in I \times N$. The Riemannian metric g_f on the regular set $C_f(N)_{\mathrm{reg}} = (0, l] \times N$ is defined as

$$g_f := dr^2 \oplus f^2(r)h \quad \text{on} \quad (0, l] \times N.$$

Then, we glue M_1 to $C_f(N)$ along their boundary N, and the resulting manifold denotes $M := M_1 \cup_N C_f(N)$. We introduce a smooth Riemannian metric g on the regular part $M_{\mathrm{reg}} = M_1 \cup_N C_f(N)_{\mathrm{reg}}$ such that g smoothly extends to M_1 from the f-horn metric g_f on $C_f(N)_{\mathrm{reg}} = (0, l] \times N$. Thus, we obtain a connected oriented compact Riemannian manifold with f-horn singularity

$$(M^m, g) = (M_1, g) \cup_N (C_f(N), g_f).$$

Then, Cheeger proved the L^2-Stokes theorem on a compact Riemannian manifold with f-horn singularity.

Theorem 2.6 (Cheeger [Ch80]). *As notations above, let $(M^m, g) = (M_1, g) \cup_N (C_f(N), g_f)$ be a connected oriented compact Riemannian manifold with f-horn singularity. Suppose that the function $f(r) = r^a$ with positive constant $a \geq 1$. Then, for a compact Riemannian manifold with r^a-horn singularity (M^m, g), the following hold:*

(1) *If* $n = 2k+1$*, the* L^2*-Stokes theorem holds for all* p*-forms on* (M, g)*;*

(2) *If* $n = 2k$*, the* L^2*-Stokes theorem holds for all* p*-forms except* $p = k$ *on* (M, g)*;*

(3) *If* $n = 2k$*, and if* $H^k(N; \mathbb{R}) = 0$*, the* L^2*-Stokes theorem holds for* k*-forms on* (M, g)*;*

(4) *If* $n = 2k$*, and if* $H^k(N; \mathbb{R}) \neq 0$*, the* L^2*-Stokes theorem does not hold for* k*-forms on* (M, g)*.*

Thus, Cheeger gave a necessary and sufficient condition that the L^2-Stokes theorem holds on a compact Riemannian manifold with r^a-horn singularity for $a \geq 1$.

Moreover, when $n = 2k$, Brüning and Lesch [BL93] p.453, Theorem 3.8, gave a choice of ideal boundary conditions. More precisely,

Theorem 2.7 (Brüning and Lesch [BL93])**.** *In the case of* $a = 1$ *as in Theorem 2.6, we have*

$$\mathrm{Dom}(d_{p,\max}) \Big/ \mathrm{Dom}(d_{p,\min}) \cong \begin{cases} H^k(N; \mathbb{R}) & \text{if } n = 2k \text{ and } p = k; \\ 0 & \text{otherwise.} \end{cases}$$

Remark 2.8. *(i) Since* $\dim H^k(N; \mathbb{R})$ *is finite, closed extensions of* $d_{p,\min}$ *are at most finite.*

(ii) In the case of more complicated singularities, Hunsicker and Mazzeo [HM05] proved the L^2*-Stokes theorem on Riemannian manifolds with edges (see [HM05] p.*3250*, Corollary* 3.11*, or [Beh09] p.*64*, Theorem* 5.11*).*

3 Warped product manifolds

We consider L^2-harmonic forms, the Ricci curvature, and the capacity of the Cauchy boundary for a general warped product Riemannian manifold.

Let (N^n, h) be a connected oriented closed Riemannian manifold of dimension n. Let $f : (0, l) \longrightarrow \mathbb{R}_+$ be a smooth positive function with $f(+0) = 0$. Suppose that $f(r)$ is the same order of r^a for some constant $0 < a < 1$, that is, there exists a positive constant $C > 0$ such that

$$C^{-1} r^a \leq f(r) \leq C r^a \quad (0 < r < l).$$

Then, we consider the warped product Riemannian manifold

$$M_f = (M^m, g) := ((0, l) \times N, \, dr^2 \oplus f(r)^2 h)$$

of dimension $m := \dim M_f = n+1$. This Riemannian manifold (M, g) is incomplete at $r = +0$. We denote by x_0 the vertex of the f-horn $C_f(N)$ corresponding to $r = 0$.

Now, we can naturally extend p-forms on N to the p-forms on $M = (0, l) \times N$: $\Omega^p(N) \subset \Omega^p(M)$.

Lemma 3.1. *For any harmonic p-form φ on (N, h), the natural extension φ on M is also a harmonic p-form on (M, g).*

Proof. First, we have $d_M \varphi = d_N \varphi = 0$ on M. Next, it is easy to see that

$$*_g(\varphi) = (-1)^p f(r)^{n-2p} dr \wedge *_h(\varphi).$$

Hence, since $d_N(*_h(\varphi)) = 0$ by the harmonicity of φ on (N, h), we have

$$\begin{aligned} d_M(*_g \varphi) &= (-1)^p\, d_M\big(\, f(r)^{n-2p} dr \wedge *_h(\varphi)\,\big) \\ &= (-1)^{p+1}\, f(r)^{n-2p} dr \wedge d_N\big(\, *_h \varphi\big) = 0. \end{aligned}$$

Therefore, we find that φ is harmonic on (M, g) □

Lemma 3.2. *If $p < \frac{1}{2}\big(n + \frac{1}{a}\big)$, then any smooth p-form φ on N naturally extends to $L^2(\Lambda^p M, g)$.*

Proof. For any $\varphi \in \Omega^p(N)$, we have

$$\begin{aligned} \|\varphi\|^2_{L^2(\Lambda^p M, g)} &= \int_0^l \int_N |\varphi|_g^2 \, d\mu_g = \int_0^l \int_N |\varphi|^2_{f^2 h} f(r)^n \, dr d\mu_h \\ &= \int_0^l f^{n-2p}(r) dr \int_N |\varphi|_h^2 \, d\mu_h \le C^{n-2p} \int_0^l r^{a(n-2p)} \, dr \|\varphi\|^2_{L^2(\Lambda^p N, h)}. \end{aligned}$$

Since $a(n - 2p) > -1$, the integral $\displaystyle\int_0^l r^{a(n-2p)}\, dr$ converges. Thus, we find $\varphi \in L^2(\Lambda^p M, g)$. □

Now, we take a cut-off function $\chi \in C^\infty(M)$ such that

$$\chi(r) := \begin{cases} 1 & \text{if } r \le \frac{l}{4}, \\ 0 & \text{if } \frac{l}{2} \le r. \end{cases}$$

If we set

$$\widetilde{\varphi} := \chi(r)\varphi \quad \text{on} \quad M = (0, l) \times N, \tag{3.1}$$

then we see that $\widetilde{\varphi} \in \Omega^p(M)$ and the support $\mathrm{supp}(\widetilde{\varphi}) \subset (0, \frac{l}{2}] \times N$.

Lemma 3.3. *For any harmonic p-form $\varphi \in \Omega^p(N)$, the p-form $\widetilde{\varphi}$ on M satisfies*

(1) $\widetilde{\varphi} \in \mathrm{Dom}(d_{p,\max})$, *if* $p < \dfrac{1}{2}\Big(n + \dfrac{1}{a}\Big)$;

(2) $f(r)^{2p-n} dr \wedge \widetilde{\varphi} \in \mathrm{Dom}(\delta_{g_{p+1,\max}})$, *if* $p > \dfrac{1}{2}\Big(n - \dfrac{1}{a}\Big)$.

Proof. (1) First, since $p < \frac{1}{2}(n + \frac{1}{a})$, by Lemma 3.2, the p-form $\widetilde{\varphi} \in \mathrm{Dom}(d_{p,\max})$ is in $L^2(\Lambda^p M, g)$. Next, since $d_N \varphi = 0$ by the harmonicity of φ on (N, h), then we have

$$d\widetilde{\varphi} = d(\chi\varphi) = d\chi \wedge \varphi + \chi d_N \varphi = \chi'(r) dr \wedge \varphi \quad \text{on} \quad \big[\tfrac{l}{4}, \tfrac{l}{2}\big] \times N.$$

Hence, since

$$\|d\widetilde{\varphi}\|^2_{L^2(\Lambda^{p+1} M, g)} = \|d\widetilde{\varphi}\|^2_{L^2(\Lambda^{p+1}[\frac{l}{4}, \frac{l}{2}] \times N, g)} < \infty,$$

we see that $d\widetilde{\varphi} \in L^2(\Lambda^{p+1} M, g)$. So, we find $\widetilde{\varphi} \in \mathrm{Dom}(d_{p,\max})$.

(2) We prove $f(r)^{2p-n} dr \wedge \widetilde{\varphi} \in \mathrm{Dom}(\delta_{g_{p+1,\max}})$, if $p > \frac{1}{2}(n - \frac{1}{a})$. It is easy to see that

$$*_g \left(f(r)^{2p-n} dr \wedge \widetilde{\varphi}\right) = *_h(\widetilde{\varphi}). \tag{3.2}$$

Since $*_h(\varphi) \in \Omega^{n-p}(N)$ and $n - p < \frac{1}{2}(n + \frac{1}{a})$, by Lemma 3.2, we see $*_h(\varphi) \in L^2(\Lambda^{n-p} M, g)$. Thus, from (3.2), it follows that

$$\|f(r)^{2p-n} dr \wedge \widetilde{\varphi}\|^2_{L^2(\Lambda^{p+1} M, g)} = \| *_h (\widetilde{\varphi})\|^2_{L^2(\Lambda^{n-p} M, g)} \leq \| *_h (\varphi)\|^2_{L^2(\Lambda^{n-p} M, g)} < \infty.$$

Hence, we see $f(r)^{2p-n} dr \wedge \widetilde{\varphi} \in L^2(\Lambda^{p+1} M, g)$.

Next, since $d_N(*_h\varphi) \equiv 0$ by the harmonicity of φ on (N, h), we have

$$d_M(*_h\widetilde{\varphi}) = d_M(\chi *_h (\varphi)) = \chi' dr \wedge (*_h\varphi). \tag{3.3}$$

Hence, from the proof of (1), it follows that

$$\begin{aligned}
\|\delta_g(f(r)^{2p-n} dr \wedge \widetilde{\varphi})\|^2_{L^2(\Lambda^p M, g)} &= \|d *_g (f(r)^{2p-n} dr \wedge \widetilde{\varphi})\|^2_{L^2(\Lambda^{m-p} M, g)} \\
&= \|d *_h (\widetilde{\varphi})\|^2_{L^2(\Lambda^{m-p} M, g)} \quad \text{(by (3.2))} \\
&= \|\chi' dr \wedge (*_h\varphi)\|^2_{L^2(\Lambda^{m-p} M, g)} \quad \text{(by (3.3))} \\
&= \|\chi' dr \wedge (*_h\varphi)\|^2_{L^2(\Lambda^{m-p}[\frac{l}{4}, \frac{l}{2}] \times N, g)} < \infty.
\end{aligned}$$

Therefore, we find $f(r)^{2p-n} dr \wedge \widetilde{\varphi} \in \mathrm{Dom}(\delta_{g_{p+1,\max}})$. □

If we make good choices of N and a, we have the following lemma.

Lemma 3.4. *If $H^p(N; \mathbb{R}) \neq 0$ for some p satisfying $\frac{1}{2}\big(n - \frac{1}{a}\big) < p < \frac{1}{2}\big(n + \frac{1}{a}\big)$, then the L^2-Stokes theorem for p-forms does not hold on (M, g).*

Proof. Since $H^p(N, \mathbb{R}) \neq 0$, by the de Rham-Hodge-Kodaira theory, there exists a non-zero harmonic p-form $\varphi \neq 0$ on N. From Lemma 3.3, it follows that $\widetilde{\varphi} \in \mathrm{Dom}(d_{\max,p})$ and that $f(r)^{2p-n} dr \wedge \widetilde{\varphi} \in \mathrm{Dom}(\delta_{g_{\max,p+1}})$. Then, by (3.2), we have

$$\widetilde{\varphi} \wedge *_g (f(r)^{2p-n} dr \wedge \widetilde{\varphi}) = \widetilde{\varphi} \wedge *_h(\widetilde{\varphi}) = \chi^2(r)|\varphi|_h^2 v_h,$$

where v_h is the volume form of (N, h). Since $\chi \equiv 1$ on $(0, \frac{l}{4}] \times N$, we have

$$\begin{aligned}\int_M d(\widetilde{\varphi} \wedge *_g (f(r)^{2p-n} dr \wedge \widetilde{\varphi})) &= \int_M d(\chi^2(r)|\varphi|_h^2 v_h) \\ &= \int_{(0,\frac{l}{4}]\times N} d(|\varphi|_h^2 v_h) + \int_{[\frac{l}{4},\frac{l}{2}]\times N} d(\chi^2(r)|\varphi|_h^2 v_h).\end{aligned}$$

Since $d(|\varphi|_h^2 v_h)$ is an $(n+1)$-form on N^n, the first term is 0. Next, by the usual Stokes theorem, the second term is

$$\begin{aligned}\int_{[\frac{l}{4},\frac{l}{2}]\times N} d(\chi^2(r)|\varphi|_h^2 v_h) &= \int_{\{\frac{l}{2}\}\times N} \chi^2(\tfrac{l}{2})|\varphi|_h^2\, v_h - \int_{\{\frac{l}{4}\}\times N} \chi^2(\tfrac{l}{4})|\varphi|_h^2\, v_h \\ &= -\int_{\{\frac{l}{4}\}\times N} |\varphi|_h^2\, v_h \quad (\text{since } \chi(\tfrac{l}{4}) = 1,\ \chi(\tfrac{l}{2}) = 0) \\ &= -\|\varphi\|^2_{L^2(\Lambda^p N, h)} \neq 0.\end{aligned}$$

Therefore, we have

$$\int_M d\big(\widetilde{\varphi} \wedge *_g (f^{2p-n}(r) dr \wedge \widetilde{\varphi})\big) \neq 0.$$

From Lemma 2.4, the L^2-Stokes theorem for p-forms does not hold on (M, g). □

Now, we recall the Ricci curvature of a warped product Riemannian manifold (M, g) (e.g. [Bes87], p.266, Proposition 9.106).

Lemma 3.5 (Ricci curvature)**.** *Let $\{e_1, \dots, e_n\}$ be a local orthonormal frame of (N^n, h). We set the local orthonormal local frame of (M, g) as $\{\widetilde{e}_0 := \frac{\partial}{\partial r}, \widetilde{e}_1 := f^{-1}e_1, \dots, \widetilde{e}_n := f^{-1}e_n\}$. Then, the Ricci operator on (M^{n+1}, g) is given by*

(1) $\mathrm{Ric}_g(\widetilde{e}_0) = -n\,\dfrac{f''(r)}{f(r)}\,\widetilde{e}_0$;

(2) $\mathrm{Ric}_g(\widetilde{e}_i) = \mathrm{Ric}_h(\widetilde{e}_i) - \left\{\dfrac{f''(r)}{f(r)} + (n-1)\left(\dfrac{f'(r)}{f(r)}\right)^2\right\}\widetilde{e}_i, \quad (i = 1, \dots, n).$

We recall the definition of the capacity of a subset (see [FOT94] **2.1** pp.64–65 or [GM13] p.612).

Definition 3.6 (capacity). *For any open subset $U \subset M$, the capacity, or 1-capacity, of U is defined as*

$$\mathrm{Cap}(U) := \inf \left\{ \|u\|^2_{H^1(M,g)} \,\middle|\, u \in H^1(M,g) \text{ and } \ u \geq 1 \text{ a.e. } U \right\},$$

where $\|u\|^2_{H^1(M,g)} = \|u\|^2_{L^2(M,g)} + \|du\|^2_{L^2(\Lambda^* M,g)}$ is the Sobolev norm of u in the Sobolev space $H^1(M,g)$. If there exist no such functions, then we define $\mathrm{Cap}(U) := \infty$. For any subset $A \subset M$, we define

$$\mathrm{Cap}(A) := \inf \left\{ \ \mathrm{Cap}(U) \,\middle|\, \text{any open subset } U \text{ with } \ A \subset U \subset M \right\}.$$

Now, we compute the capacity of the Cauchy boundary $\partial_c M := \overline{M} \setminus M = \{x_0\}$, where $\overline{M}$ is the completion as the metric space M with respect to the Riemannian distance d_g.

Lemma 3.7. *If $a \geq \frac{1}{n}$, then we have* $\mathrm{Cap}(\overline{M}_{\mathrm{sing}}) = 0$.

Proof. We take the cut-off function $\chi_\varepsilon : [0,l) \to [0,1]$ such that

$$\chi_\varepsilon(r) := \begin{cases} 1 & (0 \leq r \leq \varepsilon), \\ 1 + \dfrac{2}{\log \varepsilon} \log\left(\dfrac{r}{\varepsilon}\right) & (\varepsilon \leq r \leq \sqrt{\varepsilon}), \\ 0 & (\sqrt{\varepsilon} \leq r). \end{cases} \tag{3.4}$$

Set $\chi_\varepsilon(x) := \chi_\varepsilon(d_g(x_0,x))$ for $x \in M$. Then, $\chi_\varepsilon \in H^1(M,g)$ and $|\chi_\varepsilon| \leq 1$ on the geodesic ball of radius $\sqrt{\varepsilon} > 0$ centered at x_0.

We prove that $\|\chi_\varepsilon\|^2_{L^2(M,g)} \to 0$ as $\varepsilon \to 0$. First, it is easy to see that

$$\begin{aligned} \|\chi_\varepsilon\|^2_{L^2(M,g)} &= \int_M |\chi_\varepsilon(r)|^2 d\mu_g = \int_0^{\sqrt{\varepsilon}} |\chi_\varepsilon(r)|^2 f(r)^n dr \int_N d\mu_h \\ &\leq \int_0^{\sqrt{\varepsilon}} f(r)^n dr \ \mathrm{vol}(N,h) \leq C^n \,\mathrm{vol}(N,h) \int_0^{\sqrt{\varepsilon}} r^{na} dr \\ &\leq C^n \,\mathrm{vol}(N,h) \int_0^{\sqrt{\varepsilon}} 1\, dr \quad (\text{by } na \geq 1) \\ &= C^n \,\mathrm{vol}(N,h) \sqrt{\varepsilon} \longrightarrow 0 \quad (\text{as } \varepsilon \to 0). \end{aligned} \tag{3.5}$$

Next, we prove that $\|d\chi_\varepsilon\|^2_{L^2(\Lambda^1 M,g)} \to 0$ as $\varepsilon \to 0$. From $d\chi_\varepsilon = \chi'_\varepsilon dr$ and

$|dr|_g = 1$, it follows that $|d\chi_\varepsilon|_g^2 = |\chi'_\varepsilon dr|_g^2 = |\chi'_\varepsilon|^2$. Since $a \geq \frac{1}{n}$, we obtain

$$\begin{aligned}
\int_M |d\chi_\varepsilon|_g^2 \, d\mu_g &= \int_0^l \int_N |\chi'_\varepsilon(r)|^2 f^n(r) \, dr d\mu_h \\
&\leq C^n \operatorname{vol}(N,h) \int_\varepsilon^{\sqrt{\varepsilon}} |\chi'_\varepsilon|^2 \, r^{an} \, dr \quad (\text{ by } f(r) \leq Cr^a) \\
&= \frac{4C^n \operatorname{vol}(N,h)}{|\log \varepsilon|^2} \int_\varepsilon^{\sqrt{\varepsilon}} \left|\frac{1}{r}\right|^2 r^{an} \, dr \\
&= \frac{4C^n \operatorname{vol}(N,h)}{|\log \varepsilon|^2} \int_\varepsilon^{\sqrt{\varepsilon}} r^{an-2} dr \\
&= \frac{4C^n \operatorname{vol}(N,h)}{|\log \varepsilon|^2} \begin{cases} \dfrac{1}{an-1} \left[r^{an-1}\right]_\varepsilon^{\sqrt{\varepsilon}} & \text{if } an > 1, \\ \left[\log r\right]_\varepsilon^{\sqrt{\varepsilon}} & \text{if } an = 1 \end{cases} \\
&= 4C^n \operatorname{vol}(N,h) \begin{cases} \dfrac{1}{an-1} \cdot \dfrac{\varepsilon^{\frac{an-1}{2}} - \varepsilon^{an-1}}{|\log \varepsilon|^2} & \text{if } an > 1, \\ \dfrac{1}{2|\log \varepsilon|} & \text{if } an = 1 \end{cases} \\
&\longrightarrow 0 \quad (\text{as } \varepsilon \to 0).
\end{aligned} \tag{3.6}$$

Therefore, from (3.5) and (3.6), we find that $\operatorname{Cap}(\partial_c M) = \operatorname{Cap}(\{x_0\}) = 0$. □

4 The proof of Theorem 1.2.

Finally, we prove Theorem 1.2.

Proof of Theorem 1.2. We take an n-dimensional closed manifold (N^n, h) as the flat n-torus $(\mathbb{T}^n, h)$, where h is a flat metric on $\mathbb{T}^n$. We take the interval $I = (0, \pi)$ (i.e., $l = \pi$) and the warping function $f(r) := \sin^{1/n}(r)$, where $a := \frac{1}{n}$. Of course, this function $f(r)$ satisfies $f(r) > 0$ on $(0, \pi)$ and $f(+0) = f(-\pi) = 0$. Furthermore, there exists a positive constant $C > 0$ such that $C^{-1} r^a \leq f(r) \leq Cr^a$ on $(0, \pi)$.

Then, we consider the warped product Riemannian manifold $(M^{n+1}, g) = ((0,\pi) \times \mathbb{T}^n, dr^2 \oplus \sin^{2a}(r)h)$, which is homeomorphic to the regular set of the suspension $\Sigma(\mathbb{T}^n)$ of $\mathbb{T}^n$. This incomplete Riemannian manifold (M^{n+1}, g) is gluing two copies of the regular set $C_{\sin^a(r)}(\mathbb{T}^n)_{\mathrm{reg}}$ along their boundaries:

$$(M^{n+1}, g) = C_{\sin^a(r)}(\mathbb{T}^n)_{\mathrm{reg}} \cup_{\mathbb{T}^n} \Big(- C_{\sin^a(r)}(\mathbb{T}^n)_{\mathrm{reg}} \Big),$$

where $-$ means the opposite orientation. By means of the partition of the unity, it is enough to show the properties (1) through (4) in Theorem 1.2 on the one side horn $C_{\sin^a(r)}(\mathbb{T}^n)_{\mathrm{reg}} = ((0, \frac{\pi}{2}) \times \mathbb{T}^n, dr^2 \oplus \sin^{2a}(r)h)$.

Indeed,

(1) Since $f(r) = \sin^a(r)$ with $a = \frac{1}{n}$ and $\mathrm{Ric}_h \equiv 0$, by Lemma 3.5, we have

- $\mathrm{Ric}_g(\widetilde{e}_0, \widetilde{e}_0) = g(\mathrm{Ric}_g(\tilde{e}_0), \widetilde{e}_0) = na\left\{1 + (1-a)\dfrac{\cos^2(r)}{\sin^2(r)}\right\} \geq 1 > 0;$
- $\mathrm{Ric}_g(\widetilde{e}_i, \widetilde{e}_i) = g(\mathrm{Ric}_g(\tilde{e}_i), \widetilde{e}_i) \geq a\left\{1 + (1-na)\dfrac{\cos^2(r)}{\sin^2(r)}\right\} = \dfrac{1}{n} > 0,$ $(i = 1, \ldots, n)$.

Hence, we see that the Ricci curvature of (M, g) satisfies $\mathrm{Ric}_g \geq \dfrac{1}{n} =: K > 0$.

(2) Since $H^p(\mathbb{T}^n; \mathbb{R}) \neq 0$, by Lemmas 3.1 and 3.2, there exist non-trivial L^2 harmonic p-forms on (M, g) for all $1 \leq p \leq n-1$.

(3) In Lemma 3.4, since $a = \dfrac{1}{n}$, the range of p is $0 < p < n$. Hence, the L^2-Stokes theorem for p-forms with all $1 \leq p \leq n-1$ does not hold on (M, g).

(4) From Lemma 3.7, we see $\mathrm{Cap}(\partial_c \overline{M}) = 0$.

□

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