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p-SPECTRUM AND COLLAPSING OF CONNECTED SUMS

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Abstract. The goal of the present paper is to calculate the limit of the spectrum of the Hodge-de Rham operator under the perturbation of collapse of one part of a connected sum. It takes place in the general problem of blowing up conical singularities introduced by R. Mazzeo and J. Rowlett.

1. Introduction

It is a common problem in differential geometry to study the limit of the spectrum of Laplace type operators under singular perturbations of the metrics, especially for the Hodge-de Rham operator acting on differential forms. The first reason is the topological meaning of this operator and the fact that by singular perturbations of the metric, one can change the topology of the manifold. A general framework is the so-called *collapsing*, where the injectivity radius goes to zero while diameter and curvature are bounded. The case of loss of dimension for the limit space has been deeply studied by John Lott, following that of K. Fukaya [F87]; see especially [Lo02a, Lo02b] and the references therein. In fact, this situation appears to generalize the adiabatic limit, studied by Mazzeo and Melrose in [MM90], because in the situation of collapsing with bounded diameter and curvature to a space of lower dimension, the manifold must fiber on this space.

J. Lott can prove that the operator whose spectrum is the limit, is the Laplacian of a flat superconnection defined on a certain fiber bundle on the limit space. In [Lo04], the hypothesis on curvature is weakened by the curvature bounded only below, and he can give a lower bound for the number of small positive eigenvalues.

And what if there is no hypothesis on the curvature? Such a situation is given, for instance, by the "collapse" (in general without bound on curvature) of thin handles started in [AC95] and accomplished in [ACP09].

Here we study the perturbation when one is shrinking one part of a connected sum, which is explained by Figure 1.

More precisely, if the manifold M of dimension $(n+1)$, $n \geq 2$, is the connected sum of M_1 and M_2 around the common point p_0, endowed with Riemannian metrics g_1, g_2, then, for the collapse of one part of the connected sum, we study the dependence on $\varepsilon \to 0$ for the manifold

$$M_\varepsilon := (M_1 - B(p_0, \varepsilon)) \cup \varepsilon.(M_2 - B(p_0, 1)),$$

where $\varepsilon.(M_2 - B(p_0,1))$ means $(M_2 - B(p_0,1), \varepsilon^2 g_2)$.

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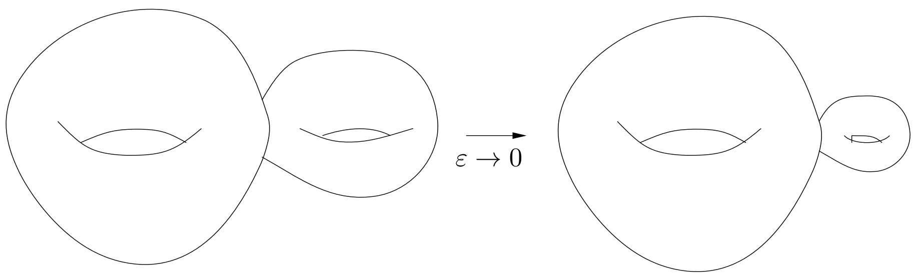

FIGURE 1. Collapsing of M_ε

To make this construction clear, we can suppose that the two metrics are flat around the point p_0. Then the boundaries of $(M_1 - B(p_0, \varepsilon), g_1)$ and $(M_2 - B(p_0, 1), \varepsilon^2 g_2)$ are isometric to the sphere $\varepsilon.\mathbb{S}^n$ and can be identified. One can then define geometrically M_ε as a piece-wise C^∞ Riemannian manifold.

This singular perturbation takes place in the general framework of resolution blowups presented in Mazzeo [Ma06]. In his terminology, M_ε is the resolution blowup of the (singular) space M_1 with *link* $\mathbb{S}^n$ and *asymptotically conical manifold* $\widetilde{M}_2$ if $\widetilde{M}_2$ is the complete manifold obtained by gluing the exterior of a ball in the Euclidean space to the boundary of $M_2 - B(p_0, 1)$. In fact, Rowlett studies, in [Ro06, Ro08], the convergence of the spectrum of a generalized Laplacian on such a situation of blowup of one isolated conical singularity (Mazzeo presents more general singularities in [Ma06]). Her result gives the convergence of the spectrum to that of an operator on M_1, but it requires a hypothesis on $\widetilde{M}_2$.

Our result is less general, applied only to the case of the Hodge-de Rham operator and in a non-singular case, but it does not require this hypothesis and the limit spectrum takes care of $\widetilde{M}_2$; see Theorem C below.

Maybe, the most interesting part of our study is that we introduce new techniques: to solve this kind of problem we have to identify a good elliptic limit problem. This means, for the M_2 part, a good boundary problem on $M_2 - B(p_0, 1)$ at the limit. It appears that, in contrast with the problem of thin handles in [AC95] or the connected sum problem studied in [T02] for functions, this boundary problem is not a kind of local but 'global': we have to introduce a condition of the Atiyah-Patodi-Singer (APS) type, as defined in [APS75].

Indeed, these APS boundary conditions are related to the Fredholm theory on the complete manifold $\widetilde{M}_2$, as explained by Carron in [C01]. Details are given below.

These techniques can be applied to study more singular situations, i.e. for general links. It will be the subject of a forthcoming work.

1.1. **The results.** As mentioned above, the manifold M, of dimension $n+1 \geq 3$ (there is no problem in dimension 2), is the connected sum of two Riemannian manifolds (M_1, g_1) and (M_2, g_2) around the common point p_0, and we suppose that the metrics are such that the boundaries of $(M_1 - B(p_0, \varepsilon), g_1)$ and $(M_2 -$

$B(p_0, 1), \varepsilon^2 g_2)$ are isometric for all ε small enough. As a consequence, (M_1, g_1) is flat in a neighborhood of p_0 and $\partial(M_2 - B(p_0, 1))$ is the standard sphere. Indeed one can write $g_1 = dr^2 + r^2 h(r)$ in the polar coordinates around $p_0 \in M_1$ and the metric $h(r)$ on the sphere converges, as $r \to 0$, to the standard metric. But if the boundaries of $(M_1 - B(p_0, \varepsilon), g_1)$ and $(M_2 - B(p_0, 1), \varepsilon^2 g_2)$ are isometric for all ε small enough, then $h(r)$ is constant for r small enough, and the conclusion follows.

One can then define geometrically $M_\varepsilon := (M_1 - B(p_0, \varepsilon)) \cup \varepsilon.(M_2 - B(p_0, 1))$ as the connected sum obtained by the collapse of M_2 (the question of the metric on M_ε is discussed below). On such a manifold, the Gauß-Bonnet operator D_ε, Sobolev spaces and also the Hodge-de Rham operator Δ_ε can be defined as follows (the details are given in [AC95]): on a manifold $X = X_1 \cup X_2$, which is the union of two Riemannian manifolds with isometric boundaries, if D_1 and D_2 are the Gauß-Bonnet "$d + d^*$" operators acting on the differential forms of each part, the quadratic form

$$q(\varphi) := \int_{X_1} |D_1(\varphi \restriction_{X_1})|^2 \, d\mu_{X_1} + \int_{X_2} |D_2(\varphi \restriction_{X_2})|^2 \, d\mu_{X_2}$$

is well-defined and closed on the domain

$$\mathrm{Dom}(q) := \{\varphi = (\varphi_1, \varphi_2) \in H^1(\Lambda T^* X_1) \times H^1(\Lambda T^* X_2) \,|\, \varphi_1 \restriction_{\partial X_1} \stackrel{L_2}{=} \varphi_2 \restriction_{\partial X_2}\},$$

where the boundary values $\varphi_i \restriction_{\partial X_i}$ are considered in the sense of the trace operator, and on this space the total Gauß-Bonnet operator $D(\varphi) = (D_1(\varphi_1), D_2(\varphi_2))$ is defined and selfadjoint. For this definition, we have, in particular, to identify $(\Lambda T^* X_1) \restriction_{\partial X_1}$ and $(\Lambda T^* X_2) \restriction_{\partial X_2}$. This can be done by decomposing the forms in tangential and normal parts (with inner normal). The equality above then means that the tangential parts are equal and the normal parts are opposite. This definition generalizes the definition in the smooth case.

The Hodge-de Rham operator $(d + d^*)^2$ of X is then defined as the operator obtained by the polarization of the quadratic form q. This gives compatibility conditions between φ_1 and φ_2 on the common boundary. We do not give details on these facts because, as remarked in the next section, it is sufficient to work with smooth metrics on M.

The multiplicity of 0 in the spectrum of Δ_ε is given by the cohomology. It is then independent of ε and can be related to the cohomology of each part by the Mayer-Vietoris argument. The point is to study the convergence of the other eigenvalues, the so-called *positive spectrum*, as $\varepsilon \to 0$. The second author has shown in [T03, Theorem 4.4, p. 21] a result of boundedness.

Proposition A ([T03])**.** *The superior limit of the k-th* positive *eigenvalue on p-forms of M_ε is bounded, as $\varepsilon \to 0$, by the k-th* positive *eigenvalue on p-forms of M_1.*

Here we show that it is also true for the lower bound. Let φ_ε be a family of eigenforms on M_ε of degree p for the Hodge-de Rham operator,

$$\Delta_\varepsilon \varphi_\varepsilon = \lambda^p(M_\varepsilon) \varphi_\varepsilon,$$

and take a subsequence ε_m, $m \in \mathbb{N}$ such that the following limit exists:

$$\lim_{m \to \infty} \lambda^p(M_{\varepsilon_m}) = \lambda^p < +\infty.$$

Theorem B. *If $\lambda^p(M_{\varepsilon_m}) \neq 0$, then $\lambda^p \neq 0$ and λ^p belongs to the spectrum of the Hodge-de Rham operator acting on the p-forms of (M_1, g_1).*

The first part is a consequence of the application of the so-called McGowan's lemma; indeed M_ε has no small eigenvalues as is shown in Proposition 1 below. To prove the convergent part of Theorem B, we shall decompose the eigenforms using the good control of the APS-boundary term. More precisely, there exist an elliptic extension $\mathcal{D}_2$ of the Gauß-Bonnet operator D_2 on $M_2(1) = M_2 - B(p_0, 1)$ and a family ψ_{ε_m} bounded in $H^1(M_1) \times \mathrm{Dom}(\mathcal{D}_2)$ such that $\|\varphi_{\varepsilon_m} - \psi_{\varepsilon_m}\| \to 0$, as $m \to \infty$.

This extension is defined by *global* boundary conditions, the conditions of APS type, in relation to the works of Carron about operators non-parabolic at infinity developed in [C01]; see Proposition 5.

We can apply this result for $\liminf_{\varepsilon\to 0} \lambda^p(M_\varepsilon)$, and if we make this construction for an orthonormal family of the first k eigenforms, we obtain, with the help of Proposition A, our main theorem.

Theorem C. *Let $M_\varepsilon = (M_1 - B(p_0, \varepsilon)) \cup \varepsilon.(M_2 - B(p_0, 1))$ be the connected sum of the two Riemannian manifolds M_1 and $\varepsilon.M_2$ of dimension $n+1$. For $p \in \{1, \dots, n\}$, let $0 < \lambda_1^p(M_1) \leq \lambda_2^p(M_1), \dots$ be the* positive *spectrum of the Hodge-de Rham operator on p-forms of M_1 and let $0 < \lambda_1^p(M_\varepsilon) \leq \lambda_2^p(M_\varepsilon), \dots$ be the* positive *spectrum of the Hodge-de Rham operator on p-forms of M_ε. Then, for all $k \geq 1$, we obtain*

$$\lim_{\varepsilon\to 0} \lambda_k^p(M_\varepsilon) = \lambda_k^p(M_1).$$

Moreover, the multiplicity of 0 is given by the cohomology and

$$H^p(M_\varepsilon; \mathbb{R}) \cong H^p(M_1; \mathbb{R}) \oplus H^p(M_2; \mathbb{R}).$$

Remark 1.A. The result of convergence of the positive spectrum is also true for $p = 0$ and has been shown in [T02]. Naturally $H^0(M_\varepsilon; \mathbb{R}) \cong H^0(M_1; \mathbb{R}) = \mathbb{R}$. By the Hodge duality this also solves the case $p = n + 1$.

1.2. **Applications.** Results on spectral convergence in singular situations can be used to give examples or counterexamples concerning possible links between spectral and geometric properties. For instance, Colbois and El Soufi have introduced in [CE03] the notion of *conformal spectrum* as the supremum, for each integer k, of the value of the k-th eigenvalue on a conformal class of metrics with fixed volume. Using the result of [T02], they could show that the conformal spectrum of a compact manifold is always bounded from below by that of the standard sphere of the same dimension.

In the same way, applying Theorem C to the case $M_1 = \mathbb{S}^{n+1}$ and $M_2 = M$, we obtain

Corollary D. *Let (M, g) be a compact Riemannian manifold of dimension $m = n + 1$, for any degree p, any integer $N \geq 1$ and any $\varepsilon > 0$. There exists on M a metric $\overline{g}$ conformal to g such that the first N positive eigenvalues on p-forms are ε-close to those of the standard sphere with the same dimension and the same volume as (M, g).*

Remark 1.B. For the completion of the panorama on this subject, let us recall that Jammes has shown, in [J07], that in dimension $m \geq 4$ the infimum of the p-spectrum in a conformal class with fixed volume is 0 for $2 \leq p \leq m-2$ and $p \neq \frac{m}{2}$, but has a positive lower bound for $p = \frac{m}{2}$.

Next, we consider the blowup of a closed complex surface M of real dimension 4. If we blowup M at one point $p \in M$, then the resulting manifold $\widehat{M}$ is also a compact complex surface which is diffeomorphic to $M \sharp \overline{\mathbb{CP}^2}$, that is, the connected sum of M and $\overline{\mathbb{CP}^2}$. Here, $\overline{\mathbb{CP}^2}$ means the reversed orientation of the complex projective space $\mathbb{CP}^2$. Note that $\widehat{M}$ is not biholomorphic to $M \sharp \overline{\mathbb{CP}^2}$.

We study the spectrum of the Hodge-de Rham operator on these manifolds. Take any Riemannian metric g_1 on M which is flat around at point p. Then, from Theorem C, we can construct a family of C^∞ metrics g_ε on $\widehat{M} \cong M \sharp \overline{\mathbb{CP}^2}$ such that the spectrum of the blowup manifold $(\widehat{M}, g_\varepsilon)$ is as close as that of the original manifold (M, g_1). In particular, all the positive spectrum of $(\widehat{M}, g_\varepsilon)$ converge to those of (M, g_1), as $\varepsilon \to 0$. This is one of the expressions of the blow-down from $(\widehat{M}, g_\varepsilon)$ to (M, g_1). Indeed, the collapsing part $\overline{\mathbb{CP}^2}$ containing an exceptional (-1)-curve $\mathbb{CP}^1$ shrinks to a point.

Another example is the *prescription of the spectrum*. This question was introduced by Colin de Verdière in [CdV86, CdV87] where he shows that he can impose any finite part of the spectrum of the Laplace-Beltrami operator on certain manifolds. To this goal, he introduced a very powerful technique of transversality and showed that this hypothesis is satisfied on certain graphs and on certain manifolds [CdV88]. The other necessary argument is a theorem of convergence. The solution of the problem of prescription, with limitation concerning multiplicity, has been given by Guerini in [G04] for the Hodge-de Rham operator, and Jammes has proven a result of prescription, without multiplicity, in a conformal class of the metric in [J08], for certain degrees of the differential forms, that is compatible with the restricted result mentioned above. In this context, our result gives, for example,

Corollary E. *Let g_0 be a metric on the sphere of dimension m. If g_0 satisfies the Strong Arnol'd Hypothesis by the terminology of* [CdV86]*, for the eigenvalue $\lambda \neq 0$ on differential forms of degree p on the sphere, then for any closed manifold M, there exists a metric such that λ belongs to the spectrum of the Hodge-de Rham operator on p-forms with the same multiplicity.*

Indeed, we take a metric g_2 on M, and for any metric g_1 close to g_0, the positive spectrum of $M_\varepsilon = \mathbb{S}^m \# \varepsilon.M$ converges, as $\varepsilon \to 0$ to the spectrum of $\mathbb{S}^m$. Then, the Strong Arnol'd Hypothesis assures that the map which associates to g_1 the *spectral quadratic form* relative to a small interval I around λ also has, for ε small enough, the matrix $\lambda \cdot \mathrm{Id}$ in its image.

Let us recall what we mean by *spectral quadratic form*. The linear space spanned by eigenforms associated with eigenvalues in the interval I is finite dimensional, and the Hodge-de Rham operator restricted to this linear space defines a quadratic form. To consider the map which associates, for each metric, this quadratic form, we must fix the space where it leaves this quadratic form. This is done by the construction of small isometries between the different eigenspaces when the metric varies. The existence of such isometries is a consequence of the convergence theorem; the details of this construction can be found in [CdV88].

This result could be used to prescribe high multiplicity for the spectrum of the Hodge-de Rham operator. Recall that Jammes obtained partial results on this subject in [J11]. His work is based on a convergence theorem (Theorem 2.8) where the limit is the Hodge-de Rham operator with absolute boundary condition on a domain. He also uses the fact that the Strong Arnol'd Hypothesis is satisfied on spheres of dimension 2, as proved in [CdV88]. It would be interesting to obtain such a result on spheres of larger dimension; the result of [CdV88] uses the conformal invariance specific to this dimension.

We now proceed to prove the theorems. Let us first describe the metrics precisely.

2. Choice of the metric

From now on, we denote

$$M_2(1) := M_2 - B(p_0, 1).$$

It is supposed here that the ball $B(p_0, 1)$ can really be embedded in the manifold M_2. This can always be satisfied by a scaling of the metric g_2 on M_2.

Recall that Dodziuk has proven in [D82, Prop. 3.3] that if two metrics g, $\overline{g}$ on the same compact manifold of n dimensions satisfy

$$\mathrm{e}^{-\eta} g \leq \overline{g} \leq \mathrm{e}^{\eta} g, \tag{2.1}$$

then the corresponding eigenvalues of the Hodge-de Rham operator acting on p-forms satisfy

$$\mathrm{e}^{-(n+2p)\eta} \lambda_k^p(g) \leq \lambda_k^p(\overline{g}) \leq \mathrm{e}^{(n+2p)\eta} \lambda_k^p(g).$$

This result is based on the fact that the multiplicity of 0 is given by the cohomology and the positive spectrum is given by exact forms, hence the min-max formula does not involve derivatives of the metric. It stays valid if one of the two metrics is only smooth by part, because in the last case the Hodge decomposition still holds true.

Then, for a metric g_1 on M_1, there exists, for each $\eta > 0$, a metric $\overline{g}_1$ on M_1 which is flat on a ball B_η centered at p_0 and such that

$$\mathrm{e}^{-\eta} g_1 \leq \overline{g}_1 \leq \mathrm{e}^{\eta} g_1.$$

Then our result can be extended to any other construction which does not suppose that the metric g_1 is flat in a neighborhood of p_0.

Now, we regard M_ε as the union of $M_1 - B(p_0, 3\varepsilon)$ and $\varepsilon.\overline{M}_2(1)$, where $\overline{M}_2(1) = \big(B_{\mathbb{R}^{n+1}}(0,3) - B_{\mathbb{R}^{n+1}}(0,1)\big) \cup M_2(1)$ is endowed with a metric only smooth by part: the Euclidean metric on the first part and the restriction of g_2 on the second part. But this metric can be approached, as close as we want, by a smooth metric which is still flat on $B_{\mathbb{R}^{n+1}}(0,3) - B_{\mathbb{R}^{n+1}}(0,\frac{3}{2})$; these two metrics will satisfy the estimate (2.1). Thus, replacing 3ε by ε for simplicity, we can suppose, without loss of generality, that we are in the following situation:

The manifold $M_2(1)$ is endowed with a metric which is conical (flat) near the boundary, namely $g_2 = ds^2 + (1-s)^2 h$, h being the canonical metric of the sphere $\mathbb{S}^n = \partial(M_2(1))$ and $s \in [0, \frac{1}{2})$ being the distance from the boundary ($M_2(1)$ looks like a trumpet) and $M_1(\varepsilon) = M_1 - B(p_0, \varepsilon)$ with a conical metric $g_1 = dr^2 + r^2 h$ around the point p_0. Thus, $M_\varepsilon = M_1(\varepsilon) \cup \varepsilon.M_2(1)$ is a smooth Riemannian manifold.

Let $\mathcal{C}_{a,b}$ be the cone $(a,b) \times \mathbb{S}^n$ endowed with the (conical) metric $dr^2 + r^2 h$.

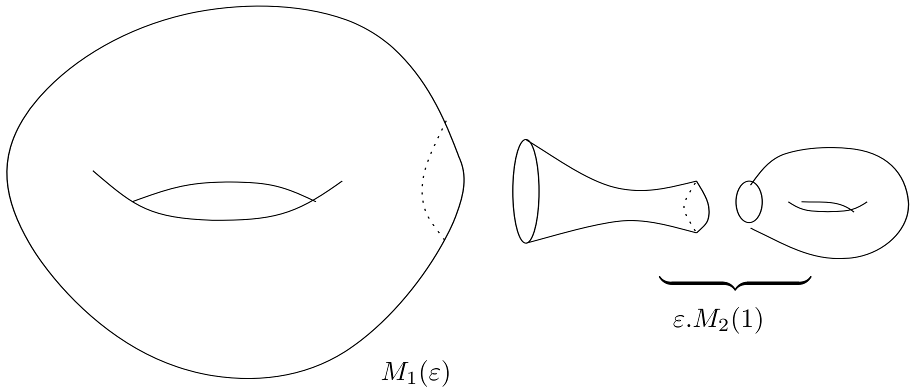

FIGURE 2. Smoothing of $(M_\varepsilon, g_\varepsilon)$

3. SMALL EIGENVALUES

Let us show that M_ε has no small eigenvalues.

Proposition 1. *For $1 \le p \le n$, there is a constant $\lambda_0 > 0$ independent of ε such that, if $\lambda^p(M_\varepsilon)$ is an eigenvalue of the Hodge-de Rham operator acting on p-forms such that $\lambda^p(M_\varepsilon) \neq 0$, then*

$$\lambda^p(M_\varepsilon) \ge \lambda_0.$$

Proof. We shall use the McGowan lemma stated in [GP95, Lemma 2.3, p. 731]. Recall that this lemma, in the spirit of the Mayer-Vietoris theorem, gives control of positive eigenvalues in terms of positive eigenvalues of certain covers with certain boundary conditions. We use the cover $M_\varepsilon = M_1(\varepsilon) \cup \varepsilon.(M_2(1) \cup \mathcal{C}_{1,2})$. Set

$$U_1 = M_1(\varepsilon) \text{ and } U_2 = \varepsilon.(M_2(1) \cup \mathcal{C}_{1,2}).$$

Then $U_{1,2} = U_1 \cap U_2 = \varepsilon.\mathcal{C}_{1,2}$ and $H^{p-1}(U_1 \cap U_2; \mathbb{R}) \cong H^{p-1}(\mathbb{S}^n; \mathbb{R}) = 0$ for $2 \le p \le n$.

Lemma 1 in [GP95] asserts that, in this case and for these values of p, the first positive eigenvalue of the Hodge-de Rham operator on exact p-forms of M_ε is bounded from below by

$$\lambda_0(\varepsilon) := \frac{1}{8}\Big\{ \Big(\frac{1}{\mu^p(U_1)} + \frac{1}{\mu^p(U_2)}\Big)\Big(\frac{2\omega_{p,n+1}\, c_\rho}{\mu^{p-1}(U_{1,2})} + 3\Big) \Big\}^{-1},$$

where $\mu^k(U)$ is the first positive eigenvalue of the Hodge-de Rham operator acting on exact k-forms on U satisfying absolute boundary conditions, $\omega_{p,n+1}$ is a combinatorial constant and c_ρ is the square of an upper bound of the first derivative of a partition of unity subordinate to the cover $\{U_1,\ U_2\}$.

In our case, c_ρ, $\mu^p(U_2)$ and $\mu^{p-1}(U_{1,2})$ are all of order ε^{-2}, but $\mu^p(U_1)$ is bounded below for $p \le n$ as was shown in [AC93] (remark that the small eigenvalue exhibited here in degree n is in the coexact spectrum). This gives a uniform bound for the exact spectrum of degree p with $2 \le p \le n$, but the exact spectrum for 1-forms comes from the spectrum on functions which has been studied in [T02]. Thus the exact spectrum is controlled for $1 \le p \le n$; by the Hodge duality it gives a control for all the positive spectrum in these degrees. Finally we can assert that there exists $\lambda_0 > 0$ such that $\lambda^p(M_\varepsilon) > \lambda_0$ for any $\varepsilon > 0$. $\square$

The proof of the main theorem, Theorem B, needs some useful notation and estimates. This is the goal of the following section.

4. Estimates and tools

From now on we suppose, given a family φ_ε of normalized eigenforms on M_ε of degree p for the Hodge-de Rham operator,

$$\Delta_\varepsilon \varphi_\varepsilon = \lambda^p(M_\varepsilon)\varphi_\varepsilon,$$

with $\limsup_{\varepsilon\to 0} \lambda^p(M_\varepsilon) = \lambda^p < +\infty$. As in [ACP09], we use the following change of variables: for φ_ε, we define $\varphi_{i,\varepsilon}$ as

$$\varphi_{1,\varepsilon} := \varphi_\varepsilon \restriction_{M_1(\varepsilon)} \quad \text{and} \quad \varphi_{2,\varepsilon} := \varepsilon^{\frac{n+1}{2}-p}\varphi_\varepsilon \restriction_{M_2(1)} . \tag{4.1}$$

We write on the cone $\mathcal{C}_{\varepsilon,1}$

$$\varphi_{1,\varepsilon} = dr \wedge r^{-(\frac{n}{2}-p+1)}\beta_{1,\varepsilon} + r^{-(\frac{n}{2}-p)}\alpha_{1,\varepsilon}$$

and define $\sigma_1(r) = (\beta_1(r), \alpha_1(r)) = U(\varphi_1)(r)$. The operator U is an isometry, and its inverse is its adjoint U^*.

On the other hand, it is more convenient to define $r := 1 - s$ for $s \in [0, \frac{1}{2}]$ and write

$$\varphi_{2,\varepsilon} = dr \wedge r^{-(\frac{n}{2}-p+1)}\beta_{2,\varepsilon} + r^{-(\frac{n}{2}-p)}\alpha_{2,\varepsilon}$$

near the boundary. Then we can define, for $r \in [\frac{1}{2}, 1]$ (the boundary of $M_2(1)$ corresponds to $r = 1$),

$$\sigma_2(r) = (\beta_2(r), \alpha_2(r)) = U(\varphi_2)(r).$$

The L^2-norm, for a p-form φ supported on M_1 in the cone $\mathcal{C}_{\varepsilon,1}$, has the expression

$$\|\varphi\|^2_{L^2(M_\varepsilon)} = \int_{\mathcal{C}_{\varepsilon,1}} |\sigma_1|^2_{g_1}\, d\mu_{g_1} + \int_{M_2(1)} |\varphi_2|^2_{g_2}\, d\mu_{g_2},$$

and the quadratic form we study is

$$\begin{aligned} q_\varepsilon(\varphi) &= \int_{M_\varepsilon} |(d+d^*)\varphi|^2_{g_\varepsilon}\, d\mu_{g_\varepsilon} \\ &= \int_{M_1(\varepsilon)} |UD_1U^*(\sigma_1)|^2\, d\mu_{g_1} + \frac{1}{\varepsilon^2}\int_{M_2(1)} |D_2(\varphi_2)|^2_{g_2}\, d\mu_{g_2}, \end{aligned} \tag{4.2}$$

where D_1, resp. D_2, is the Gauß-Bonnet operator of M_1, resp. M_2, namely $D_j = d + d^*$ acting on differential forms. In terms of σ_1, which, a priori, belongs to $C^\infty([\varepsilon,1); C^\infty(\Lambda^{p-1}T^*\mathbb{S}^n) \oplus C^\infty(\Lambda^pT^*\mathbb{S}^n))$, the operator has, on the cone of M_1, the expression

$$UD_1U^* = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \left(\partial_r + \frac{1}{r}A\right) \text{ with } A = \begin{pmatrix} \frac{n}{2} - P & -D_0 \\ -D_0 & P - \frac{n}{2} \end{pmatrix},$$

where P is the operator of degree which multiplies by p a p-form and D_0 is the Gauß-Bonnet operator of the sphere $\mathbb{S}^n$.

The Hodge-de Rham operator has, in these coordinates, the expression

$$U\Delta_1U^* = -\partial_r^2 + \frac{1}{r^2}A(A+1).$$

The same expressions are valid for UD_2U^* and $U\Delta_2U^*$ near the boundary of $M_2(1)$, but we shall not use them, since we need global estimates on this part.

The compatibility condition is, for the quadratic form, $\varepsilon^{\frac{1}{2}}\alpha_1(\varepsilon) = \alpha_2(1)$ and $\varepsilon^{\frac{1}{2}}\beta_1(\varepsilon) = \beta_2(1)$ or

$$\sigma_2(1) = \varepsilon^{\frac{1}{2}}\sigma_1(\varepsilon). \tag{4.3}$$

The compatibility condition for the Hodge-de Rham operator, of the first order, is obtained by expressing that $D\varphi \sim (UD_1U^*\sigma_1, \varepsilon^{-1}UD_2U^*\sigma_2)$ belongs to the domain of D. In terms of σ it gives

$$\sigma_2'(1) = \varepsilon^{\frac{3}{2}}\sigma_1'(\varepsilon). \tag{4.4}$$

Let ξ_1 be the cut-off function on M_1 around p_0:

$$\xi_1(r) := \begin{cases} 1 & \text{if } 0 \le r \le \frac{1}{2}, \\ 0 & \text{if } 1 \le r. \end{cases}$$

Proposition 2. *For our given family φ_ε satisfying $\Delta\varphi_\varepsilon = \lambda^p(M_\varepsilon)\varphi_\varepsilon$ with $\lambda^p(M_\varepsilon)$ bounded, the family $(1-\xi_1).\varphi_{1,\varepsilon}$ is bounded in $H^1(\Lambda^p M_1, g_1)$.*

Then it remains to study $\xi_1.\varphi_{1,\varepsilon}$, which can be expressed with the polar coordinates. This is the goal of the next section.

Remark 3. The same cannot be done with the component on M_2, or more precisely this does not give what we want to prove, namely that this component goes to 0 with ε. To do so we first have to consider $\varphi_{2,\varepsilon}$ in the domain of an elliptic operator. This is the main difficulty, in contrast with the case of the functions. In fact, we will decompose $\varphi_{2,\varepsilon}$ in a part which clearly goes to 0 and another part which belongs to the domain of an elliptic operator. This operator is naturally D_2, but the point is to determine the boundary conditions.

4.1. **Expression of the quadratic form.** For any φ such that the component φ_1 is supported in the cone $\mathcal{C}_{1,\varepsilon}$, one has, with $\sigma_1 = U\varphi_1$ and by the same calculations as in [ACP09],

$$\begin{aligned}
&\int_{\mathcal{C}_{\varepsilon,1}} |D_1\varphi|^2_{g_1} d\mu_{g_1} = \int_\varepsilon^1 \left\| \Big(\partial_r + \frac{1}{r}A\Big)\sigma_1 \right\|^2_{L^2(\mathbb{S}^n)} dr \\
&= \int_\varepsilon^1 \Big[\|\sigma_1'\|^2_{L^2(\mathbb{S}^n)} + \frac{2}{r}\big(\sigma_1', A\sigma_1\big)_{L^2(\mathbb{S}^n)} + \frac{1}{r^2}\|A\sigma_1\|^2_{L^2(\mathbb{S}^n)} \Big] dr \\
&= \int_\varepsilon^1 \Big[\|\sigma_1'\|^2_{L^2(\mathbb{S}^n)} + \partial_r\Big\{\frac{1}{r}\big(\sigma_1, A\sigma_1\big)_{L^2(\mathbb{S}^n)}\Big\} + \frac{1}{r^2}\Big\{\big(\sigma_1, A\sigma_1\big)_{L^2(\mathbb{S}^n)} + \|A\sigma_1\|^2_{L^2(\mathbb{S}^n)}\Big\}\Big] dr \\
&= \int_\varepsilon^1 \Big[\|\sigma_1'\|^2_{L^2(\mathbb{S}^n)} + \frac{1}{r^2}\big(\sigma_1, (A+A^2)\sigma_1\big)_{L^2(\mathbb{S}^n)} \Big] dr - \frac{1}{\varepsilon}\big(\sigma_1(\varepsilon), A\sigma_1(\varepsilon)\big)_{L^2(\mathbb{S}^n)},
\end{aligned}$$

where we remark $\sigma_1(1) = 0$ and ${}^tA = A$. We then have

$$\begin{aligned}
q_\varepsilon(\varphi) = \int_\varepsilon^1 \Big[\|\sigma_1'\|^2_{L^2(\mathbb{S}^n)} + \frac{1}{r^2}\big(\sigma_1, (A+A^2)\sigma_1\big)_{L^2(\mathbb{S}^n)} \Big] dr - \frac{1}{\varepsilon}\big(\sigma_1(\varepsilon), A\sigma_1(\varepsilon)\big)_{L^2(\mathbb{S}^n)} \\
+ \frac{1}{\varepsilon^2}\int_{M_2(1)} |D_2\varphi_2|^2_{g_2}\, d\mu_{g_2}.
\end{aligned} \tag{4.5}$$

On the other hand we have, as well,

$$\begin{aligned}\int_{\mathcal{C}_{\frac{1}{2},1}} |D_2\varphi|^2_{g_2} d\mu_{g_2} &= \int_{\frac{1}{2}}^1 \Big\| \Big(\partial_r + \frac{1}{r}A\Big)\sigma_2 \Big\|^2_{L^2(\mathbb{S}^n)} dr \\ &= \int_{\frac{1}{2}}^1 \Big[\|\sigma_2'\|^2_{L^2(\mathbb{S}^n)} + \frac{1}{r^2}\big(\sigma_2,\,(A+A^2)\sigma_2\big)_{L^2(\mathbb{S}^n)} \Big] dr \\ &\qquad + \big(\sigma_2(1),\, A\sigma_2(1)\big)_{L^2(\mathbb{S}^n)} - \big(\sigma_2(\tfrac{1}{2}),\, A\sigma_2(\tfrac{1}{2})\big)_{L^2(\mathbb{S}^n)}.\end{aligned}$$

Thus the first boundary terms annihilate, because of the compatibility condition (4.3) $\sigma_2(1) = \varepsilon^{\frac{1}{2}}\sigma_1(\varepsilon)$, and one also has

$$\begin{aligned}(4.6)\quad q_\varepsilon(\varphi) &= \int_\varepsilon^1 \Big[\|\sigma_1'\|^2_{L^2(\mathbb{S}^n)} + \frac{1}{r^2}(\sigma_1, (A+A^2)\sigma_1)_{L^2(\mathbb{S}^n)} \Big] dr \\ &+ \frac{1}{\varepsilon^2}\int_{\frac{1}{2}}^1 \Big[\|\sigma_2'\|^2_{L^2(\mathbb{S}^n)} + \frac{1}{r^2}(\sigma_2,\,(A+A^2)\sigma_2)_{L^2(\mathbb{S}^n)} \Big] dr - \frac{1}{\varepsilon^2}\big(\sigma_2(\tfrac{1}{2}),\, A\sigma_2(\tfrac{1}{2})\big)_{L^2(\mathbb{S}^n)}.\end{aligned}$$

We remark that the boundary term $-(\sigma_2(\frac{1}{2}), A\,\sigma_2(\frac{1}{2}))_{L^2(\mathbb{S}^n)}$ is positive if σ_2 belongs to the eigenspace of A with negative eigenvalues. In fact we know the spectrum of A:

4.2. **Spectrum of A.** The spectrum of A was calculated in Brüning and Seeley [BS88, p. 703]. By their result, it holds that the spectrum of A is given by the values

$$\gamma = \frac{(-1)^{p+1}}{2} \pm \sqrt{\mu^2 + \Big(\frac{n-1}{2} - p\Big)^2},$$

where $0 \le p \le n$ and μ^2 runs over the spectrum of the Hodge-de Rham operator on the standard sphere $\Delta_{\mathbb{S}^n}$ acting on the coclosed p-forms. Since the spectrum of the standard sphere $\mathbb{S}^n$ was calculated in [GM75, p. 283] we see that $\mu^2 = (p+k+1)(n-p+k)$ $(k = 0,1,2,\dots)$ if $1 \le p \le n$, or if $p = 0$ and μ^2 is in the coexact spectrum. Hence, $\mu^2 \ge (n-p)(p+1)$ and we have

$$\begin{aligned}\mu^2 + \Big(\frac{n-1}{2} - p\Big)^2 &\ge (n-p)(p+1) + \Big(\frac{n+1}{2} - (p+1)\Big)^2 \\ &\ge \Big(\frac{n+1}{2}\Big)^2.\end{aligned}$$

So any eigenvalue γ of A satisfies

$$(4.7)\qquad |\gamma| \ge \frac{n}{2}.$$

For $p = 0$, the eigenvalues of A corresponding to the constant function are in fact $\pm\frac{n}{2}$ as we can see with the expression of A, so the lower bound (4.7) is always valid and, in particular, $0 \notin \mathrm{Spec}(A)$.

Consequence. The elliptic operator $A(A+1)$ is non-negative (and positive if $n \ge 3$). Indeed $A(A+1) = (A+\frac{1}{2})^2 - \frac{1}{4}$, and the values of the eigenvalues of A give the conclusion.

4.3. **Equations satisfied.** On the cones, $\sigma = (\sigma_1, \sigma_2)$ satisfies the equations

$$\Big(- \partial_r^2 + \frac{1}{r^2} A(A+1) \Big) \sigma_1 = \lambda_\varepsilon \sigma_1,$$
$$\Delta_2 U^* \sigma_2 = \varepsilon^2 \lambda_\varepsilon U^* \sigma_2,$$

and the compatibility conditions have been given in (4.3) and (4.4):

$$\sigma_2(1) = \varepsilon^{\frac{1}{2}} \sigma_1(\varepsilon), \quad \sigma_2'(1) = \varepsilon^{\frac{3}{2}} \sigma_1'(\varepsilon).$$

We decompose σ_1 into a base of eigenvectors of A:

$$\sigma_1 = \sum_{\gamma \in \mathrm{Spec}(A)} \sigma_1^\gamma \quad \text{and} \quad A\sigma_1^\gamma = \gamma \sigma_1^\gamma.$$

4.4. **Boundary control.** We know that $\displaystyle\int_\varepsilon^1 \big\| (\partial_r + \frac{1}{r} A)\, \sigma_1 \big\|^2_{L^2(\mathbb{S}^n)}\, dr \leq \lambda^p + 1$ for ε small enough, since $\lambda^p(M_\varepsilon)$ has an upper bound by Proposition A and the expression of D_1, (4.1). This inequality stays valid for $\xi_1 \sigma_1$ with a larger constant: there exists a constant $\Lambda > 0$ such that for any $\varepsilon > 0$

$$\sum_{\gamma \in \mathrm{Spec}(A)} \int_\varepsilon^1 \Big\| \partial_r(\xi_1 \sigma_1^\gamma) + \frac{\gamma}{r} (\xi_1 \sigma_1^\gamma) \Big\|^2_{L^2(\mathbb{S}^n)} dr \leq \Lambda. \tag{4.8}$$

We remark that $\partial_r \sigma + \dfrac{\gamma}{r}\sigma = r^{-\gamma} \partial_r (r^\gamma \sigma)$. Since $\gamma < 0$ implies $\gamma \leq (-\frac{n}{2})$, by $\xi_1(1) = 0$, (4.8) and the Schwarz inequality, we have

$$\begin{aligned}
\| \varepsilon^\gamma \sigma_1^\gamma(\varepsilon) \|^2_{L^2(\mathbb{S}^n)} &= \Big\| \int_\varepsilon^1 \partial_r (r^\gamma \xi_1 \sigma_1^\gamma)\, dr \Big\|^2_{L^2(\mathbb{S}^n)} \\
&\leq \Big| \int_\varepsilon^1 r^{2\gamma}\, dr \Big| \cdot \int_\varepsilon^1 \Big\| \partial_r(\xi_1 \sigma_1^\gamma) + \frac{\gamma}{r} (\xi_1\, \sigma_1^\gamma) \Big\|^2_{L^2(\mathbb{S}^n)} dr \\
&\leq \Lambda \frac{\varepsilon^{2\gamma+1}}{|2\gamma+1|},
\end{aligned} \tag{4.9}$$

where the last inequality follows from $\gamma \leq (-\frac{n}{2})$. Thus we see $\sigma_1^\gamma(\varepsilon) = O(\varepsilon^{\frac{1}{2}} / \sqrt{|2\gamma + 1|})$. This suggests that the limit σ is harmonic on $M_2(1)$ with the boundary condition $\Pi_{<0} \sigma_2 = 0$, where $\Pi_{<0}$ (resp. $\Pi_{>0}$) denotes the spectral projection of A onto the total eigenspace of negative (resp. positive) eigenvalues. The limit problem appearing here is a boundary condition of the Atiyah-Patodi-Singer type [APS75]. Indeed we have

Proposition 4. *There exists a constant $C > 0$ such that the boundary value satisfies, for all $\varepsilon > 0$,*

$$\| \Pi_{<0} (\sigma_{1,\varepsilon}(\varepsilon)) \|^2_{L^2(\mathbb{S}^n)} \leq C \varepsilon.$$

Proof. We know that $q(\xi_1 \varphi_{1,\varepsilon}, \varphi_{2,\varepsilon})$ is bounded by Λ. On the other hand, the expression of the quadratic form (4.5) can be done with respect to the decomposition into $\mathrm{Im}(\Pi_{>0})$ and $\mathrm{Im}(\Pi_{<0})$. Namely,

$$\begin{aligned}
q_\varepsilon(\xi_1\varphi_{1,\varepsilon},\,\varphi_{2,\varepsilon}) &= \int_\varepsilon^1 \left\|\Big(\partial_r+\frac{1}{r}A\Big)\Pi_{<0}(\xi_1\sigma_{1,\varepsilon})\right\|^2_{L^2(\mathbb{S}^n)}dr \\
&\quad + \int_\varepsilon^1 \left\|\Big(\partial_r+\frac{1}{r}A\Big)\Pi_{>0}(\xi_1\sigma_{1,\varepsilon})\right\|^2_{L^2(\mathbb{S}^n)}dr + \frac{1}{\varepsilon^2}\|D_2\varphi_2\|^2_{L^2(M_2(1),g_2)} \\
&\ge \int_\varepsilon^1 \left\|\Big(\partial_r+\frac{1}{r}A\Big)\Pi_{<0}(\xi_1\sigma_{1,\varepsilon})\right\|^2_{L^2(\mathbb{S}^n)}dr \\
&\ge \int_\varepsilon^1 \left\{\|\Pi_{<0}(\xi_1\sigma_{1,\varepsilon})'\|^2_{L^2(\mathbb{S}^n)} + \frac{1}{r^2}\Big(\Pi_{<0}(\xi_1\sigma_{1,\varepsilon}),\,(A+A^2)\Pi_{<0}(\xi_1\sigma_{1,\varepsilon})\Big)_{L^2(\mathbb{S}^n)}\right\}dr \\
&\quad - \frac{1}{\varepsilon}\big(\Pi_{<0}\sigma_1(\varepsilon),\,A\circ\Pi_{<0}\sigma_1(\varepsilon)\big)_{L^2(\mathbb{S}^n)} \\
&\ge \frac{n}{2\varepsilon}\,\|\,\Pi_{<0}\sigma_{1,\varepsilon}(\varepsilon)\,\|^2_{L^2(\mathbb{S}^n)},
\end{aligned}$$

since $A(A+1)$ is non-negative and $(-A\circ\Pi_{<0})\ge\frac{n}{2}$. □

4.5. **Limit problem.** Here we study good candidates for the limit Gauß-Bonnet operator. On M_1 the problem is clear; the question here is to identify the boundary conditions on $M_2(1)$.

• On M_1, the natural problem is the Friedrich extension of D_1 on the cone. It is not a real conical singularity, and $\Delta_1 = D_1^*\circ D_1$ is the usual Hodge-de Rham operator.

• For $n\ge 2$, the forms on $M_2(1)$ satisfying $D_2\varphi=0$ and $\Pi_{<0}\circ U(\varphi)=0$ on the boundary are precisely the L^2-forms in $\mathrm{Ker}(D_2)$ on the large manifold $(\widetilde{M}_2,\tilde g_2)$ obtained from $(M_2(1),g_2)$ by gluing a conical cylinder $[1,\infty)\times\mathbb{S}^n$ with the metric dr^2+r^2h, i.e. the exterior of the sphere in $\mathbb{R}^{n+1}$.

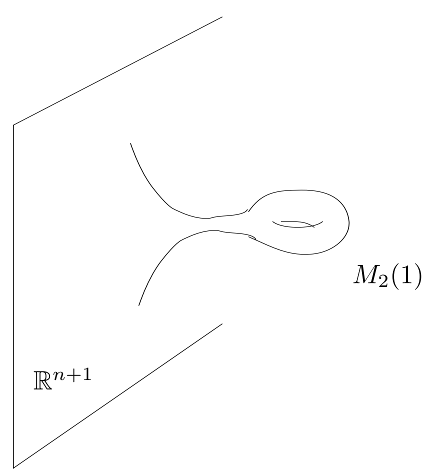

FIGURE 3. $(\widetilde{M}_2,\tilde g_2)$

Indeed, these L^2-forms on $(\widetilde{M}_2,\tilde g_2)$ must satisfy $(\partial_r+\frac{1}{r}A)\sigma=0$, i.e. for any $\gamma\in\mathrm{Spec}(A)$, there exists $\sigma_0^\gamma\in\mathrm{Ker}(A-\gamma)$ such that $\sigma^\gamma=r^{-\gamma}\sigma_0^\gamma\in L^2(\widetilde{M}_2,\tilde g_2)$, which is possible only for $\gamma>\frac{1}{2}$. This limit problem is of the category *non-parabolic* *at infinity* in the terminology due to Carron [C01]; see particularly Theorem 2.1 there. Then as a consequence of Theorem 0.4 in [C01], we know that its kernel is of finite dimension. More precisely it gives:

Proposition 5. *The operator D_2 acting on the forms of $M_2(1)$, with the boundary condition $\Pi_{<0} \circ U = 0$, is elliptic in the sense that the H^1-norm of elements of the domain is controlled by the norm of the graph. Namely, there exists a constant $C > 0$ such that any $\varphi \in H^1(\Lambda^p M_2(1), g_2)$ with $\Pi_{<0} \circ U(\varphi) = 0$ satisfies*

$$\|\varphi\|^2_{H^1(M_2(1),g_2)} \leq C \;\{\; \|\varphi\|^2_{L^2(M_2(1),g_2)} + \|D_2\varphi\|^2_{L^2(M_2(1),g_2)} \;\}.$$

We denote by $\mathcal{D}_2$ this operator.

Corollary 6. *The kernel of $\mathcal{D}_2$ is of finite dimension and can be identified with a subspace of the total space $\displaystyle\sum_{p=0}^{n+1} H^p(M_2(1);\mathbb{R})$ of the absolute cohomology.*

We shall see in Corollary 16 below that this kernel is in fact the total space $\displaystyle\sum_{p=1}^{n} H^p(M_2;\mathbb{R})$.

Proof. We show that there exists a constant $C > 0$ such that for each $\varphi \in H^1(\Lambda^p M_2(1), g_2)$ satisfying $\Pi_{<0} \circ U(\varphi) = 0$, then

$$\|\varphi\|^2_{H^1(M_2(1))} \leq C \;\big\{\|\varphi\|^2_{L^2(M_2(1))} + \|D_2\varphi\|^2_{L^2(M_2(1))}\big\}.$$

Thus $\mathcal{D}_2$ is a closable operator.

Denote, for such a φ, by $\tilde\varphi$ its harmonic prolongation on $\widetilde{M}_2 - M_2(1) = \mathbb{R}^{n+1} - B(0,1)$. Then $\tilde\varphi$ is in the domain of the Dirac operator on $(\widetilde{M}_2, \tilde g_2)$, which is elliptic. This means that for each smooth function f on $\widetilde{M}_2$ with compact support, there exists a constant $C_f > 0$ such that for any $\psi \in \mathrm{Dom}(D_2)$,

$$\|f.\psi\|^2_{H^1(\widetilde{M}_2)} \leq C_f \{\; \|\psi\|^2_{L^2(\widetilde{M}_2)} + \|D_2\psi\|^2_{L^2(\widetilde{M}_2)} \;\}$$

(it is the fact that an operator 'non-parabolic at infinity' is continuous from its domain to H^1_{loc}, Theorem 1.2 of Carron [C01]).

If we apply this inequality to some $f = 1$ on $M_2(1)$ and $\psi = \tilde\varphi$, we obtain in particular that

$$\|\varphi\|^2_{H^1(M_2(1))} \leq C\{\; \|\tilde\varphi\|^2_{L^2(\widetilde{M}_2)} + \|D_2\tilde\varphi\|^2_{L^2(\widetilde{M}_2)} \;\}$$

with $C = C_f$. Since $D_2\tilde\varphi \equiv 0$ on $\widetilde{M}_2 - M_2(1) = \mathbb{R}^{n+1} - B(0,1)$, we see that

$$\|D_2\tilde\varphi\|^2_{L^2(\widetilde{M}_2)} = \|D_2\varphi\|^2_{L^2(M_2(1))}.$$

Now we can write, by the use of cut-off functions, $\varphi = \varphi_0 + \bar\varphi$, where φ_0 vanishes near the boundary and $\bar\varphi$ is supported in $\frac{1}{2} \leq r \leq 1$. Then $\tilde\varphi_0 = 0$ outside of $M_2(1)$. So, for the control of $\|\tilde\varphi\|_{L^2(\widetilde{M}_2)}$, we can suppose that $\varphi = \bar\varphi$. We write $U\varphi = \sigma$ and $\sigma = \sum_\gamma \sigma^\gamma$ on the eigenspaces of A.

Since $\sigma = U(\tilde\varphi)$ is harmonic on $\mathbb{R}^{n+1} - B(0,1)$, it follows that for any γ

$$\sigma^\gamma(r) = r^{-\gamma}\sigma^\gamma(1).$$

Thus we have

$$\begin{aligned}
\|\tilde{\varphi}\|^2_{L^2(\mathbb{R}^{n+1}-B(0,1))} &= \|U(\tilde{\varphi})\|^2_{L^2(\mathbb{R}^{n+1}-B(0,1))} = \|\sigma\|^2_{L^2([1,\infty)\times\mathbb{S}^n)} \\
&= \|\sum_{\gamma>0}\sigma^\gamma(r)\|^2_{L^2([1,\infty)\times\mathbb{S}^n)} = \|\sum_{\gamma>0} r^{-\gamma}\sigma^\gamma(1)\|^2_{L^2([1,\infty)\times\mathbb{S}^n)} \\
&= \sum_{\gamma>0}\int_1^\infty r^{-2\gamma}dt\cdot\|\sigma^\gamma(1)\|^2_{L^2(\mathbb{S}^n)} \\
&= \sum_{\gamma>0}\frac{1}{2\gamma-1}\|\sigma^\gamma(1)\|^2_{L^2(\mathbb{S}^n)}.
\end{aligned}$$

Now we remark that $\gamma\geq 1,\ \sigma^\gamma(\frac{1}{2})=0$ and $\sigma^\gamma(1)=\displaystyle\int_{\frac{1}{2}}^1\partial_r(r^\gamma\sigma^\gamma)dr$. By the Cauchy-Schwarz inequality, we have

$$\begin{aligned}
\|\sigma^\gamma(1)\|^2_{L^2(\mathbb{S}^n)} &= \Big\|\int_{\frac{1}{2}}^1 r^\gamma\cdot r^{-\gamma}\partial_r(r^\gamma\sigma^\gamma)dr\Big\|^2_{L^2(\mathbb{S}^n)} \\
&\leq \int_{\frac{1}{2}}^1\|r^{-\gamma}\partial_r(r^\gamma\sigma^\gamma)\|^2_{L^2(\mathbb{S}^n)}dr\cdot\int_{\frac{1}{2}}^1 r^{2\gamma}dr \\
&\leq \frac{1}{2\gamma+1}\,\|UD_2U^*\sigma^\gamma\|^2_{L^2([\frac{1}{2},1]\times\mathbb{S}^n)}.
\end{aligned}$$

As a consequence,

$$\begin{aligned}
\sum_{\gamma>0}\frac{1}{2\gamma-1}\|\sigma^\gamma(1)\|^2_{L^2(\mathbb{S}^n)} &\leq \sum_{\gamma>0}\frac{1}{4\gamma^2-1}\,\|UD_2U^*\sigma^\gamma\|^2_{L^2([\frac{1}{2},1]\times\mathbb{S}^n)} \\
&\leq \|\,D_2\varphi\,\|^2_{L^2(M_2(1))}.
\end{aligned}$$

Hence, changing the constant, we also have

$$\|\varphi\|^2_{H^1(M_2(1),g_2)}\leq C\,\big\{\,\|\varphi\|^2_{L^2(M_2(1),g_2)}+\|D_2\varphi\|^2_{L^2(M_2(1),g_2)}\,\big\}. \qquad \square$$

Alternative proof of Proposition 5*, in the spirit of* [APS75]. In order to study this boundary condition, it is better to again write the p-form near the boundary as $\varphi_2 = dr\wedge r^{-(n/2-p+1)}\beta_2 + r^{-(n/2-p)}\alpha_2$ with, as before, $U(\varphi_2)=\sigma_2=(\beta_2,\alpha_2)$. On the cone $r\in[\frac{1}{2},1]$, $UD_2U^*=\partial_r+\frac{1}{r}A$ and we can construct, as in [APS75], a parametrix of D_2 by gluing an interior parametrix with one constructed on the 'long' cone $r\in(0,1]$ as follows:

Given a form ψ on $M_2(1)$, if we look for a form φ such that $D_2\varphi=\psi$, we write ψ as the sum of two terms, the first one with support in the neighborhood of the boundary and the second one vanishing near the boundary. On the second term, we apply an interior parametrix Q_0 of the elliptic operator D_2. Let us now suppose that φ is supported in the cone $r\in[\frac{1}{2},1]$. We decompose $U\psi$ into the eigenspaces of $A: U\psi=\sum_\gamma\psi^\gamma$, and also if $U\varphi=\sum_\gamma\varphi^\gamma$, then φ^γ must satisfy

$$\partial_r\varphi^\gamma+\frac{\gamma}{r}\varphi^\gamma = r^{-\gamma}\partial_r(r^\gamma\varphi^\gamma)=\psi^\gamma.$$

We take the solution

$$\varphi^\gamma = r^{-\gamma}\int_1^r \rho^\gamma \psi^\gamma(\rho)\,d\rho \quad \text{if } \gamma < 0,$$
$$\varphi^\gamma = r^{-\gamma}\int_0^r \rho^\gamma \psi^\gamma(\rho)\,d\rho \quad \text{if } \gamma > 0.$$

Then $\gamma < 0$ implies $\varphi^\gamma(1) = 0$. It is now easy to verify that $\mathcal{D}_2$ satisfies the *singular elliptic* property (SE) of [Le97, p. 54] (with $\rho(x) = \sqrt{x}$).

This fact and the property $\mathrm{Spec}(A) \cap (-1, +1) = \emptyset$ assure the construction of the parametrix on the cone. Here we refer to Büning and Seeley [BS88] who make this construction and also to Lesch [Le97] for more general settings. In fact, this parametrix on the cone $r \in (0, 1]$ gives only H^1-regularity with weight function (as described in [Le97, Proposition 1.3.12] and the following), but the construction of a parametrix on $M_2(1)$ is made by gluing these two parametrixes with the help of cut-off functions. This means that the region near the singular point $r = 0$ of the cone will be cut off, and we stay in a region where the weight function is controlled by constants from above and from below. Thus, we finally obtain control of the usual H^1-norm.

4.6. **Boundedness.** Recall that $A(A + 1)$ is non-negative.

We define the cut-off function χ supported in $[\frac{3}{4}, 1)$ as

$$\chi(r) := \begin{cases} 0 & \text{if } 0 \le r \le \frac{3}{4}, \\ 1 & \text{if } \frac{7}{8} \le r \le 1. \end{cases}$$

Proposition 7. *Again denoting $\sigma_{2,\varepsilon} := U(\varphi_{2,\varepsilon})$, the family of p-forms,*

$$\psi_{2,\varepsilon} := \varphi_{2,\varepsilon} - U^*\big(\,\Pi_{<0}(\chi\sigma_{2,\varepsilon})\,\big),$$

belongs to the domain of $\mathcal{D}_2$ and satisfies the following:

(1) $\{\psi_{2,\varepsilon}\}_{\varepsilon>0}$ *is uniformly bounded in $H^1(M_2(1), g_2)$;*

(2) $\lim_{\varepsilon\to 0} \|\psi_{2,\varepsilon} - \varphi_{2,\varepsilon}\|_{L^2(M_2(1),g_2)} = 0$;

(3) $\lim_{\varepsilon\to 0} \|D_2(\psi_{2,\varepsilon} - \varphi_{2,\varepsilon})\|_{L^2(M_2(1),g_2)} = 0.$

Remark 8. As a consequence of Proposition 7, there exists a subsequence of $\varphi_{2,\varepsilon}$, which converges in L^2 to a harmonic form satisfying the boundary condition of $\mathcal{D}_2$. That is, there exists $\psi_2 \in \mathrm{Ker}(\mathcal{D}_2)$ such that

$$\varphi_{2,\varepsilon} \to \psi_2 \in \mathrm{Ker}(\mathcal{D}_2) \;\text{ weakly in } H^1(M_2(1), g_2) \;\text{ as } \varepsilon \to 0.$$

Proof. (1) It is clear that $\psi_{2,\varepsilon}$ belongs to the domain of $\mathcal{D}_2$ and is a bounded family for the operator norm. Thus, by Proposition 5, it is also a bounded family in $H^1(M_2(1), g_2)$.

(2) We have

$$\|\psi_{2,\varepsilon} - \varphi_{2,\varepsilon}\|^2_{L^2(M_2(1),g_2)} \le \int_{\frac{3}{4}}^1 \|\,\Pi_{<0}\sigma_{2,\varepsilon}(r)\,\|^2_{L^2(\mathbb{S}^n)}\,dr, \tag{4.10}$$

but as a consequence of (4.6)

$$
\begin{aligned}
&\| \Pi_{<0}\sigma_{2,\varepsilon}(r) \|^2_{L^2(\mathbb{S}^n)} = \int_{\frac12}^r \partial_t \big\{ \| \Pi_{<0}\sigma_{2,\varepsilon}(t) \|^2_{L^2(\mathbb{S}^n)} \big\}\, dt + \| \Pi_{<0}\sigma_{2,\varepsilon}(\tfrac12)\|^2_{L^2(\mathbb{S}^n)} \\
&= 2\int_{\frac12}^r \big(\Pi_{<0}\sigma'_{2,\varepsilon}(t),\, \Pi_{<0}\sigma_{2,\varepsilon}(t) \big)_{L^2(\mathbb{S}^n)}\, dt + \| \Pi_{<0}\sigma_{2,\varepsilon}(\tfrac12)\|^2_{L^2(\mathbb{S}^n)} \\
&\leq 2\Big\{ \int_{\frac12}^r \|\Pi_{<0}\sigma'_{2,\varepsilon}(t) \|^2_{L^2(\mathbb{S}^n)}\, dt \Big\}^{\frac12} \cdot \|\Pi_{<0}\sigma_{2,\varepsilon}\|_{L^2([\frac12,r]\times\mathbb{S}^n)} + \| \Pi_{<0}(\sigma_{2,\varepsilon}(\tfrac12))\|^2_{L^2(\mathbb{S}^n)} \\
&\leq 2\varepsilon\Lambda + \varepsilon^2 \frac{2}{n}\Lambda,
\end{aligned}
$$

by using the Cauchy-Schwarz inequality, the fact that the L^2-norm of φ_ε is 1 and that $(-A\circ\Pi_{<0}) \geq \frac{n}{2}$.

(3) Finally, we prove $\lim_{\varepsilon\to 0} \|D_2(\psi_{2,\varepsilon} - \varphi_{2,\varepsilon})\|_{L^2(M_2(1),g_2)} = 0$. Since $\operatorname{supp}(\chi) \subset [\frac34, 1]$, we see that

$$
\begin{aligned}
\| D_2(\varphi_{2,\varepsilon} - \psi_{2,\varepsilon}) \|_{L^2(M_2(1),g_2)} &= \| D_2\big(U^*\circ \Pi_{<0}(\chi\sigma_{2,\varepsilon})\big) \|_{L^2(M_2(1),g_2)} \\
&\leq 2C \,\| \Pi_{<0}(\sigma_{2,\varepsilon})\|_{L^2([\frac34,1]\times\mathbb{S}^n)} + \| D_2\big(U^*\circ\Pi_{<0}(\sigma_{2,\varepsilon})\big) \|_{L^2(\mathcal{C}_{\frac34,1})},
\end{aligned} \tag{4.11}
$$

where $C > 0$ is a constant depending only on χ'.

Now, since equation (4.10) converges to 0 by the result (2), the first term of (4.11) converges to 0:

$$
\| \Pi_{<0}(\sigma_{2,\varepsilon})\|^2_{L^2([\frac34,1]\times\mathbb{S}^n)} = \int_{\frac34}^1 \| \Pi_{<0}(\sigma_{2,\varepsilon})\|^2_{L^2(\mathbb{S}^n)}\, dr \to 0 \quad (\varepsilon\to 0).
$$

The second term of (4.11) also converges to 0, since

$$
\begin{aligned}
&\| D_2\big(U^*\circ\Pi_{<0}(\sigma_{2,\varepsilon})\big) \|^2_{L^2(\mathcal{C}_{\frac34,1})} \\
&\qquad\leq \| D_2\big(U^*\circ\Pi_{<0}(\sigma_{2,\varepsilon})\big) \|^2_{L^2(\mathcal{C}_{\frac34,1})} + \| D_2\big(U^*\circ\Pi_{>0}(\sigma_{2,\varepsilon})\big) \|^2_{L^2(\mathcal{C}_{\frac34,1})} \\
&\qquad\leq \| D_2\varphi_{2,\varepsilon} \|^2_{L^2(\mathcal{C}_{\frac34,1})} \leq \| D_2\varphi_{2,\varepsilon} \|^2_{L^2(M_2(1),g_2)} \\
&\qquad\leq \varepsilon^2 \| D_\varepsilon\varphi_\varepsilon \|^2_{L^2(M_\varepsilon)} = \varepsilon^2\lambda^{(p)}(M_\varepsilon) \leq O(\varepsilon^2).
\end{aligned}
$$

Therefore, we see that $\lim_{\varepsilon\to 0} \| D_2(\psi_{2,\varepsilon} - \varphi_{2,\varepsilon}) \|_{L^2(M_2(1),g_2)} = 0$. □

Since the trace operator from $H^1(M_2(1))$ to $H^{\frac12}(\partial M_2(1))$ is bounded, we have

Corollary 9. *The family $\{\Pi_{>0}\sigma_{2,\varepsilon}(1)\}$ is bounded in $H^{\frac12}(\mathbb{S}^n) = H^{\frac12}(\partial M_2(1))$ as the boundary value of $\psi_{2,\varepsilon}$.*

We now define a better prolongation of $\Pi_{>0}\sigma_2(1)$ to $M_1(\varepsilon)$. More generally, we set

$$
\begin{aligned}
P_\varepsilon : \Pi_{>0}\Big(H^{\frac12}(\mathbb{S}^n)\Big) &\to H^1(\mathcal{C}_{\varepsilon,1}) \\
\sigma = \sum_{\substack{\gamma\in\operatorname{Spec}(A)\\ \gamma>0}} \sigma_\gamma &\mapsto P_\varepsilon(\sigma) := \sum_{\substack{\gamma\in\operatorname{Spec}(A)\\ \gamma>0}} \varepsilon^{\gamma-\frac12} r^{-\gamma}\sigma_\gamma.
\end{aligned}
$$

This is a harmonic prolongation, since each $\varepsilon^{\gamma-\frac{1}{2}}\, r^{-\gamma}\sigma_\gamma$ is harmonic on the cone $\mathcal{C}_{\varepsilon,1}$. The exponent $(-\frac{1}{2})$ over ε comes from the compatibility condition (4.3). We remark that there exists a constant $C > 0$ such that

$$\| P_\varepsilon(\sigma) \|^2_{L^2(\mathcal{C}_{\varepsilon,1})} \le C \sum_{\gamma>0} \|\sigma_\gamma\|^2_{L^2(\mathbb{S}^n)} = C \| \Pi_{>0}\sigma_{2,\varepsilon}(1) \|^2_{L^2(\mathbb{S}^n)}. \tag{4.12}$$

If $\psi_{2,\varepsilon} \in \mathrm{Dom}(\mathcal{D}_2)$ for the same cut-off function ξ_1 whose value is 1 for $0 \le r \le \frac{1}{2}$ and 0 for $r \ge 1$, then $\big(\xi_1 P_\varepsilon(\psi_{2,\varepsilon} \restriction_{\partial M_2(1)}),\, \psi_{2,\varepsilon}\big)$ defines through the isometry U an element of $H^1(\Lambda^p M_\varepsilon, g_\varepsilon)$. Set

$$\tilde{\psi}_{1,\varepsilon} := \xi_1 \cdot P_\varepsilon(\psi_{2,\varepsilon} \restriction_{\partial M_2(1)}).$$

We now decompose $\varphi_{1,\varepsilon}$ on the cone $\mathcal{C}_{\varepsilon,1}$ as follows:

$$\varphi_{1,\varepsilon} = \varphi^+_{1,\varepsilon} + \varphi^-_{1,\varepsilon}, \tag{4.13}$$

according to the decomposition of $\sigma_{1,\varepsilon} = U(\varphi_{1,\varepsilon})$ into the positive and negative spectrum of A. Then $\tilde{\psi}_1$ and φ^+_ε have the same values on the boundary. So the difference $(\varphi^+_\varepsilon - \tilde{\psi}_{1,\varepsilon})$ can be viewed in $H^1(\Lambda^p M_1, g_1)$ by a prolongation to 0 on the ε-ball, while the boundary value of $\varphi^-_{1,\varepsilon}$ is small. We introduce the cut-off function taken in [ACP09]:

$$\xi_\varepsilon(r) := \begin{cases} 0 & \text{if } \ r \le 2\varepsilon, \\ \dfrac{\log(2\varepsilon) - \log r}{\log(\sqrt{\varepsilon})} & \text{if } \ 2\varepsilon \le r \le 2\sqrt{\varepsilon}, \\ 1 & \text{if } \ 2\sqrt{\varepsilon} \le r. \end{cases}$$

Lemma 10.

$$\lim_{\varepsilon\to 0} \| (1-\xi_\varepsilon)\cdot(\xi_1\, \varphi^-_{1,\varepsilon}) \|_{L^2(M_1(\varepsilon), g_1)} = 0.$$

This follows from the estimates in Proposition 4.

Proposition 11. *The p-forms* $\psi_{1,\varepsilon} := (1-\xi_1)\varphi_{1,\varepsilon} + (\xi_1\varphi^+_{1,\varepsilon} - \tilde{\psi}_{1,\varepsilon}) + \xi_\varepsilon\, \xi_1\, \varphi^-_{1,\varepsilon}$ *belong to* $H^1(\Lambda^p M_1, g_1)$ *and define a bounded family.*

Proof. We will show that each term is bounded. For the first one, $(1-\xi_1)\varphi_{1,\varepsilon}$ can be defined on M_1 by a prolongation to 0. So, it is already proven in Proposition 2. For the second one, we remark that

$$f_\varepsilon := \Big(\partial_r + \frac{A}{r}\Big)(\xi_1\varphi^+_{1,\varepsilon} - \tilde{\psi}_{1,\varepsilon}) = \Big(\partial_r + \frac{A}{r}\Big)(\xi_1\varphi^+_{1,\varepsilon}) - \partial_r(\xi_1)\, P_\varepsilon(\psi_{2,\varepsilon} \restriction_{\partial M_2(1)}). \tag{4.14}$$

Then f_ε is uniformly bounded in $L^2(M_1, g_1)$ because of (4.12). This estimate also shows that the L^2-norm of $(\xi_1\varphi^+_{1,\varepsilon} - \tilde{\psi}_{1,\varepsilon})$ is bounded. Thus the family $\{\xi_1\varphi^+_{1,\varepsilon} - \tilde{\psi}_{1,\varepsilon}\}$ is bounded for the q-norm on (M_1, g_1), which is equivalent to the H^1-norm.

For the third one, we use the estimate due to the expression of the quadratic form. Since $\displaystyle\int_{\mathcal{C}_{r,1}} |D_1(\xi_1\varphi^-_{1,\varepsilon})|^2_{g_1}\, d\mu_{g_1} \le \Lambda$, the boundary term is estimated by

$$-\frac{1}{r}\big(\sigma^-_{1,\varepsilon}(r),\, A\, \sigma^-_{1,\varepsilon}(r)\big)_{L^2(\mathbb{S}^n)} \le \Lambda, \tag{4.15}$$

by the same argument as in Proposition 4. Now

$$\begin{aligned} \| D_1(\xi_\varepsilon\xi_1\varphi^-_{1,\varepsilon}) \|_{L^2(M_1)} &\le \| \xi_\varepsilon\, D_1(\xi_1\varphi^-_{1,\varepsilon}) \|_{L^2(M_1)} + \| \, |d\xi_\varepsilon|\, \xi_1\varphi^-_{1,\varepsilon} \|_{L^2(M_1)} \\ &\le \| D_1(\xi_1\varphi^-_{1,\varepsilon}) \|_{L^2(\mathcal{C}_{\varepsilon,1})} + \| \, |d\xi_\varepsilon|\, \varphi^-_{1,\varepsilon} \|_{L^2(\mathcal{C}_{\varepsilon,1})}\, . \end{aligned}$$

The first term is bounded and, with $(-A\circ\Pi_{<0})\geq\frac{n}{2}$ and the estimate (4.15), we have

$$\|\,|d\xi_\varepsilon|\,\varphi_{1,\varepsilon}^-\|^2_{L^2(\mathcal{C}_{\varepsilon,1})}\leq\frac{8\Lambda}{n\,|\log\varepsilon|^2}\int_{2\varepsilon}^{2\sqrt{\varepsilon}}\frac{dr}{r}\leq\frac{4\Lambda}{n\,|\log\varepsilon|}.$$

This completes the proof. □

In fact, the decomposition used here is almost orthogonal:

Lemma 12.

$$\lim_{\varepsilon\to 0}(\,\varphi_{1,\varepsilon}-\tilde\psi_{1,\varepsilon},\tilde\psi_{1,\varepsilon}\,)_{L^2(M_1(\varepsilon),g_1)}=0.$$

Proof of Lemma 12. If we decompose the terms into the eigenspaces of A, then $\tilde\psi_{1,\varepsilon}$ involves only the positive eigenvalues of A. So, for the decomposition $\varphi_{1,\varepsilon}=\varphi_{1,\varepsilon}^++\varphi_{1,\varepsilon}^-$, we see that

$$(\,\varphi_{1,\varepsilon}^-,\tilde\psi_{1,\varepsilon}\,)_{L^2(\mathcal{C}_{\varepsilon,1})}=0.$$

Hence we have only to prove that

$$\lim_{\varepsilon\to 0}(\,\varphi_{1,\varepsilon}^+-\tilde\psi_{1,\varepsilon},\,\tilde\psi_{1,\varepsilon})_{L^2(\mathcal{C}_{\varepsilon,1})}=0.$$

We now set $f_\varepsilon=\sum_{\gamma>0}f^\gamma$ and $\varphi_{1,\varepsilon}^+-\tilde\psi_{1,\varepsilon}=\sum_{\gamma>0}\varphi_0^\gamma$. Equation (4.14) and the fact that $(\varphi_{1,\varepsilon}^+-\tilde\psi_{1,\varepsilon})(\varepsilon)=0$ give

$$\varphi_0^\gamma(r)=r^{-\gamma}\int_\varepsilon^r\rho^\gamma f^\gamma(\rho)\,d\rho.$$

Then for each positive eigenvalue $\gamma>\frac{3}{2}$ of A, by using integration by parts and the Schwarz inequality, we have

$$\begin{aligned}
(\varphi_0^\gamma,\,\tilde\psi_1^\gamma)_{L^2(\mathcal{C}_{\varepsilon,1})}&=\Big(r^{-\gamma}\int_\varepsilon^r\rho^\gamma f^\gamma(\rho)\,d\rho,\,\xi_1\,\varepsilon^{\gamma-\frac12}r^{-\gamma}\sigma_{2,\varepsilon}^\gamma(1)\Big)_{L^2(\mathcal{C}_{\varepsilon,1})}\\
&=\varepsilon^{\gamma-\frac12}\int_\varepsilon^1 r^{-2\gamma}\xi_1\Big\{\int_\varepsilon^r\rho^\gamma\left(f^\gamma(\rho),\,\sigma_{2,\varepsilon}^\gamma(1)\right)_{L^2(\mathbb{S}^n)}d\rho\Big\}\,dr\\
&\leq\varepsilon^{\gamma-\frac12}\,\|\sigma_{2,\varepsilon}^\gamma\|_{L^2(\mathbb{S}^n)}\int_\varepsilon^1 r^{-2\gamma}dr\int_\varepsilon^r\rho^\gamma\|f^\gamma(\rho)\|_{L^2(\mathbb{S}^n)}\,d\rho\\
&=\varepsilon^{\gamma-\frac12}\,\|\sigma_{2,\varepsilon}^\gamma\|_{L^2(\mathbb{S}^n)}\Big\{\Big[\frac{r^{1-2\gamma}}{1-2\gamma}\int_\varepsilon^r\rho^\gamma\,\|f^\gamma(\rho)\|_{L^2(\mathbb{S}^n)}\,d\rho\Big]_{r=\varepsilon}^1\\
&\qquad-\int_\varepsilon^1\frac{r^{1-2\gamma}}{1-2\gamma}\,r^\gamma\,\|f^\gamma(r)\|_{L^2(\mathbb{S}^n)}\,dr\Big\}\\
&\leq\frac{\varepsilon^{\gamma-\frac12}}{2\gamma-1}\,\|\sigma_{2,\varepsilon}^\gamma\|_{L^2(\mathbb{S}^n)}\int_\varepsilon^1 r^{-\gamma+1}\,\|f^\gamma(r)\|_{L^2(\mathbb{S}^n)}\,dr\\
&\leq\frac{\varepsilon^{\gamma-\frac12}}{2\gamma-1}\,\|\sigma_{2,\varepsilon}^\gamma\|_{L^2(\mathbb{S}^n)}\Big\{\int_\varepsilon^1 r^{2-2\gamma}dr\Big\}^{\frac12}\cdot\Big\{\int_\varepsilon^1\|f^\gamma(r)\|^2_{L^2(\mathbb{S}^n)}\,dr\Big\}^{\frac12}\\
&\leq\frac{\varepsilon^{\gamma-\frac12}}{2\gamma-1}\,\|\sigma_{2,\varepsilon}^\gamma\|_{L^2(\mathbb{S}^n)}\,\frac{\varepsilon^{(-\gamma+\frac32)}}{\sqrt{2\gamma-3}}\,\|f^\gamma\|_{L^2(\mathcal{C}_{\varepsilon,1})}\\
&\leq C\,\varepsilon\,\|\sigma_{2,\varepsilon}^\gamma\|_{L^2(\mathbb{S}^n)}\,\|f^\gamma\|_{L^2(\mathcal{C}_{\varepsilon,1})}\leq O(\varepsilon).
\end{aligned}$$

From Corollary 9 and Proposition 11, we proved Lemma 12 in the case of $\gamma>\frac{3}{2}$.

By the previous calculus of the spectrum of A, Section 4.2, we know that the possible values $\gamma \leq \frac{3}{2}$ are only $\frac{3}{2}$ and 1.

By the same calculations as above, we can obtain similar estimates: for $\gamma = \frac{3}{2}$, the order of the bound is $\varepsilon\sqrt{|\log(\varepsilon)|}$, and for $\gamma = 1$ it is $\sqrt{\varepsilon}$. □

5. Proof of Theorem B

Recall that φ_ε is a family of eigenforms on M_ε of degree p for the Hodge-de Rham operator: $\Delta_\varepsilon\varphi_\varepsilon = \lambda^p(M_\varepsilon)\varphi_\varepsilon$, the family $\lambda^p(M_\varepsilon)$ is bounded and ε_m, $m \in \mathbb{N}$, is a sequence converging to zero such that the following limit exists:

$$\lim_{m\to\infty} \lambda^p(M_{\varepsilon_m}) = \lambda^p < +\infty.$$

It is the case, for instance, if $\lambda^p = \liminf_{\varepsilon\to 0} \lambda^p(M_\varepsilon)$.

Lemma 13. *If $\lambda^p \neq 0$, then $\lambda^p(M_{\varepsilon_m}) \neq 0$ for all $\varepsilon_m > 0$, and*

$$(1) \qquad \lim_{m\to\infty} \| \tilde{\psi}_{1,\varepsilon_m} \|_{L^2(M_1(\varepsilon_m),g_1)} = 0,$$

$$(2) \qquad \lim_{m\to\infty} \|\psi_{2,\varepsilon_m}\|_{L^2(M_2(1),g_2)} = 0,$$

and also $\lim_{m\to\infty} \left\{ \|D_1(\tilde{\psi}_{1,\varepsilon_m})\|_{L^2(M_1(\varepsilon_m),g_1)} + \| D_2(\psi_{2,\varepsilon_m})\|_{L^2(M_2(1),g_2)} \right\} = 0.$

Proof. We know, by Proposition 1, that there exists a universal lower bound for positive eigenvalues on M_ε, so if $\lambda^p = \lim_{m\to\infty} \lambda^p(M_{\varepsilon_m})$ is positive, then it means that all the $\lambda^p(M_{\varepsilon_m})$ are also positive!

Next, for any normalized eigen p-form φ_ε associated with the positive eigenvalue $\lambda^p(M_\varepsilon) > 0$, we define

$$\psi_\varepsilon := (\tilde{\psi}_{1,\varepsilon}, \psi_{2,\varepsilon}) \in \mathrm{Dom}(q_\varepsilon). \tag{5.1}$$

Note that ψ_ε satisfies the compatibility condition (4.3), $\varepsilon^{\frac{1}{2}}\, \tilde{\psi}_{1,\varepsilon}(\varepsilon) = \psi_{2,\varepsilon}(1)$.

Since the dimension of $\mathrm{Ker}(\mathcal{D}_2)$ is finite by Corollary 6, we can decompose $\psi_{2,\varepsilon} \in \mathrm{Dom}(\mathcal{D}_2)$ into

$$\psi_{2,\varepsilon} = \psi^0_{2,\varepsilon} + \psi^\perp_{2,\varepsilon}, \tag{5.2}$$

where $\psi^0_{2,\varepsilon} \in \mathrm{Ker}(\mathcal{D}_2)$ and $\psi^\perp_{2,\varepsilon}$ is its orthogonal part in $\mathrm{Dom}(\mathcal{D}_2)$. In fact, since $\psi_{2,\varepsilon}$ and $\psi^0_{2,\varepsilon}$ satisfy the boundary condition, then $\psi^\perp_{2,\varepsilon} = \psi_{2,\varepsilon} - \psi^0_{2,\varepsilon}$ also satisfies the boundary condition $\Pi_- \circ U = 0$.

Now, we can prolong them onto the cone $\mathcal{C}_{\varepsilon,1}$ by means of P_ε, that is,

$$\begin{cases} \tilde{\psi}^0_{1,\varepsilon} := \xi_1\, P_\varepsilon(\psi^0_{2,\varepsilon} \restriction_{\partial M_2(1)}), \\ \tilde{\psi}^\perp_{1,\varepsilon} := \xi_1\, P_\varepsilon(\psi^\perp_{2,\varepsilon} \restriction_{\partial M_2(1)}). \end{cases}$$

So, we set

$$\psi_\varepsilon := (\tilde{\psi}_{1,\varepsilon},\, \psi_{2,\varepsilon}) \in \mathrm{Dom}(q_\varepsilon),$$

and also set

$$\begin{cases} \psi^0_\varepsilon := (\tilde{\psi}^0_{1,\varepsilon},\, \psi^0_{2,\varepsilon}) \in \mathrm{Dom}(q_\varepsilon), \\ \psi^\perp_\varepsilon := (\tilde{\psi}^\perp_{1,\varepsilon},\, \psi^\perp_{2,\varepsilon}) \in \mathrm{Dom}(q_\varepsilon). \end{cases} \tag{5.3}$$

Since $D_2\psi^0_{2,\varepsilon} \equiv 0$ and $D_1(P_\varepsilon(\sigma^0_{2,\varepsilon}(1))) \equiv 0$, we obtain that

$$\begin{aligned}
q_\varepsilon(\psi^0_\varepsilon) &= q_1(\tilde{\psi}^0_{1,\varepsilon}) + \frac{1}{\varepsilon^2} q_2(\psi^0_{2,\varepsilon}) \\
&= \| D_1(\tilde{\psi}^0_{1,\varepsilon}) \|^2_{L^2(\mathcal{C}_{\varepsilon,1})} = \int_{\frac{1}{2}}^1 \| \xi_1' \, P_\varepsilon(\sigma^0_{2,\varepsilon}(1)) \|^2_{L^2(\mathbb{S}^n)} dr \\
&\le C \int_{\frac{1}{2}}^1 \| \sum_{\gamma>0} \varepsilon^{\gamma-\frac{1}{2}} r^{-\gamma} \, \sigma^{0,\gamma}_{2,\varepsilon}(1) \|^2_{L^2(\mathbb{S}^n)} dr \\
&\le C \int_{\frac{1}{2}}^1 \sum_{\gamma>0} \varepsilon^{2\gamma-1} r^{-2\gamma} \| \, \sigma^{0,\gamma}_{2,\varepsilon}(1) \|^2_{L^2(\mathbb{S}^n)} dr \\
&\le C\varepsilon^{n-1} \| \, \sigma^0_{2,\varepsilon}(1) \|^2_{L^2(\mathbb{S}^n)} \int_{\frac{1}{2}}^1 r^{-n} dr \quad (\text{by } \gamma \ge \frac{n}{2}).
\end{aligned}$$

We remark here that the boundary term $\| \, \sigma^0_{2,\varepsilon_m}(1) \|^2_{L^2(\mathbb{S}^n)}$ is uniformly bounded because of Proposition 4 and the compatibility condition, so

$$q_{\varepsilon_m}(\psi^0_{\varepsilon_m}) = O(\varepsilon_m^{n-1}) \le O(\varepsilon_m). \tag{5.4}$$

Then, we see that

$$(\varphi_{\varepsilon_m}, \, \psi^0_{\varepsilon_m})_{L^2(M_{\varepsilon_m}, g_{\varepsilon_m})} = O(\sqrt{\varepsilon_m}). \tag{5.5}$$

In fact, since the eigenvalue $\lambda^p(M_{\varepsilon_m})$ has a positive lower bound by Proposition 1 and the estimate (5.4) holds, we see that

$$\begin{aligned}
\lambda^p(M_{\varepsilon_m}) \, (\varphi_{\varepsilon_m}, \, \psi^0_{\varepsilon_m})_{L^2(M_{\varepsilon_m})} &= (\Delta_{\varepsilon_m} \varphi_{\varepsilon_m}, \, \psi^0_{\varepsilon_m})_{L^2(M_{\varepsilon_m})} = q_{\varepsilon_m}(\varphi_{\varepsilon_m}, \, \psi^0_{\varepsilon_m}) \\
&\le \sqrt{q_{\varepsilon_m}(\varphi_{\varepsilon_m})} \cdot \sqrt{q_{\varepsilon_m}(\psi^0_{\varepsilon_m})} \\
&\le \sqrt{\lambda^p(M_{\varepsilon_m})} \cdot O(\sqrt{\varepsilon_m}).
\end{aligned}$$

So we have obtained

$$(\varphi_{\varepsilon_m}, \, \psi^0_{\varepsilon_m})_{L^2(M_{\varepsilon_m})} \le \frac{O(\sqrt{\varepsilon_m})}{\sqrt{\lambda^p(M_{\varepsilon_m})}}.$$

On the other hand, $\| \, \mathcal{D}_2(\psi^\perp_{2,\varepsilon_m}) \|^2_{L^2(M_2(1))} = O(\varepsilon_m)$ implies $\| \, \psi^\perp_{2,\varepsilon_m} \|^2_{L^2(M_2(1))} = O(\sqrt{\varepsilon_m})$. In fact, this follows from the min-max type inequality for $\mathcal{D}_2$. More precisely, we set the constant

$$\mu_1 := \inf_{\psi \ne 0 \in \mathrm{Ker}(\mathcal{D}_2)^\perp} \frac{q_2(\psi)}{\|\psi\|^2_{L^2(M_2(1), g_2)}}.$$

Then we have that $\mu_1 > 0$ by using Proposition 5. Thus, from Proposition 7(3), it follows that for $\psi^\perp_{2,\varepsilon_m} \in \mathrm{Ker}(\mathcal{D}_2)^\perp$,

$$\begin{aligned}
0 &< \|\psi^\perp_{2,\varepsilon_m} \|^2_{L^2(M_2(1), g_2)} \le \frac{1}{\mu_1} \, q_2(\psi^\perp_{2,\varepsilon_m}) \le \frac{1}{\mu_1} \, q_2(\psi_{2,\varepsilon_m}) \\
&\le \frac{2}{\mu_1} \left\{ \|D_2 \varphi_{2,\varepsilon_m}\|^2_{L^2(M_2(1), g_2)} + \|\mathcal{D}_2(\psi_{2,\varepsilon_m} - \varphi_{2,\varepsilon_m}) \|^2_{L^2(M_2(1), g_2)} \right\} \\
&\le \frac{2}{\mu_1} \left\{ O({\varepsilon_m}^2) + O(\varepsilon_m) \right\} \le O(\varepsilon_m).
\end{aligned}$$

From the continuity of P_ε and the ellipticity of $\mathcal{D}_2$, we also obtain that

$$\| \tilde{\psi}^{\perp}_{1,\varepsilon_m} \|^2_{L^2(M_1(\varepsilon_m),g_1)} = O(\varepsilon_m).$$

Thus, we obtain

$$\|\psi^{\perp}_{\varepsilon_m}\|^2_{L^2(M_{\varepsilon_m},g_{\varepsilon_m})} = \|\tilde{\psi}^{\perp}_{1,\varepsilon_m}\|^2_{L^2(M_1(\varepsilon_m),g_1)} + \|\psi^{\perp}_{2,\varepsilon_m}\|^2_{L^2(M_2(1),g_2)} = O(\varepsilon_m). \tag{5.6}$$

From (5.5) and (5.6), it follows that

$$\begin{aligned}(\varphi_{\varepsilon_m}, \psi_{\varepsilon_m})_{L^2(M_{\varepsilon_m},g_{\varepsilon_m})} &= (\varphi_{\varepsilon_m}, \psi_{\varepsilon_m} - \psi^0_{\varepsilon_m})_{L^2(M_{\varepsilon_m})} + (\varphi_{\varepsilon_m}, \psi^0_{\varepsilon_m})_{L^2(M_{\varepsilon_m})} \\ &= (\varphi_{\varepsilon_m}, \psi^{\perp}_{\varepsilon_m})_{L^2(M_{\varepsilon_m})} + O(\sqrt{\varepsilon_m}) \\ &\le \|\varphi_{\varepsilon_m}\|_{L^2(M_{\varepsilon_m})} \cdot \|\psi^{\perp}_{\varepsilon_m}\|_{L^2(M_{\varepsilon_m})} + O(\sqrt{\varepsilon_m}) \\ &\le 1 \cdot \sqrt{O(\varepsilon_m)} + O(\sqrt{\varepsilon_m}) = O(\sqrt{\varepsilon_m}).\end{aligned} \tag{5.7}$$

Next, we estimate the inner product:

$$\begin{aligned}(\varphi_{\varepsilon_m} - \psi_{\varepsilon_m}, \psi_{\varepsilon_m})_{L^2(M_{\varepsilon_m})} = (\varphi_{1,\varepsilon_m} - \tilde{\psi}_{1,\varepsilon_m}, \tilde{\psi}_{1,\varepsilon_m})_{L^2(M_1(\varepsilon_m),g_1)} \\ + (\varphi_{2,\varepsilon_m} - \psi_{2,\varepsilon_m}, \psi_{2,\varepsilon_m})_{L^2(M_2(1),g_2)}.\end{aligned} \tag{5.8}$$

From Lemma 12, the first term is

$$\begin{aligned}(\varphi_{1,\varepsilon_m} - \tilde{\psi}_{1,\varepsilon_m}, \tilde{\psi}_{1,\varepsilon_m})_{L^2(M_1(\varepsilon_m),g_1)} &= (\varphi_{1,\varepsilon_m} - \tilde{\psi}_{1,\varepsilon_m}, \tilde{\psi}_{1,\varepsilon_m})_{L^2(\mathcal{C}_{\varepsilon_m,1})} \\ &= (\varphi^{+}_{1,\varepsilon_m} - \tilde{\psi}_{1,\varepsilon_m}, \tilde{\psi}_{1,\varepsilon_m})_{L^2(\mathcal{C}_{\varepsilon_m,1})} + (\varphi^{-}_{1,\varepsilon_m}, \tilde{\psi}_{1,\varepsilon_m})_{L^2(\mathcal{C}_{\varepsilon_m,1})} \\ &= (\varphi^{+}_{1,\varepsilon_m} - \tilde{\psi}_{1,\varepsilon_m}, \tilde{\psi}_{1,\varepsilon_m})_{L^2(\mathcal{C}_{\varepsilon_m,1})} \to 0 \quad (\varepsilon_m \to 0).\end{aligned} \tag{5.9}$$

From Proposition 7(3), the second term of (5.8) is

$$\begin{aligned}(\varphi_{2,\varepsilon_m} - \psi_{2,\varepsilon_m}, \psi_{2,\varepsilon_m})_{L^2(M_2(1),g_2)} \le \|\varphi_{2,\varepsilon_m} - \psi_{2,\varepsilon_m}\|_{L^2(M_2(1),g_2)} \\ \cdot \|\psi_{2,\varepsilon_m}\|_{L^2(M_2(1),g_2)} \to 0.\end{aligned} \tag{5.10}$$

Therefore, from equations (5.9), (5.10) and (5.7), we see that

$$\begin{aligned}\|\psi_{\varepsilon_m}\|^2_{L^2(M_{\varepsilon_m})} &= (\psi_{\varepsilon_m} - \varphi_{\varepsilon_m}, \psi_{\varepsilon_m})_{L^2(M_{\varepsilon_m})} + (\varphi_{\varepsilon_m}, \psi_{\varepsilon_m})_{L^2(M_{\varepsilon_m})} \\ &= -(\varphi_{1,\varepsilon_m} - \tilde{\psi}_{1,\varepsilon_m}, \tilde{\psi}_{1,\varepsilon_m})_{L^2(\mathcal{C}_{\varepsilon_m,1})} \\ &\quad - (\varphi_{2,\varepsilon_m} - \psi_{2,\varepsilon_m}, \psi_{2,\varepsilon_m})_{L^2(M_2(1),g_2)} \\ &\quad + (\varphi_{\varepsilon_m}, \psi_{\varepsilon_m})_{L^2(M_{\varepsilon_m})} \to 0 \qquad (\varepsilon_m \to 0).\end{aligned}$$

Hence, we conclude that

$$0 = \lim_{m\to\infty} \|\psi_{\varepsilon_m}\|^2_{L^2(M_{\varepsilon_m})} = \lim_{m\to\infty} \left\{ \|\tilde{\psi}_{1,\varepsilon_m}\|^2_{L^2(M_1(\varepsilon_m),g_1)} + \|\psi_{2,\varepsilon_m}\|^2_{L^2(M_2(1),g_2)} \right\}.$$

Thus, we have finished the proof of Lemma 13. □

As a consequence of Proposition 7 and Lemma 13, we obtain

Corollary 14. $\lim_{m\to\infty} \|\varphi_{2,\varepsilon_m}\|_{L^2(M_2(1),g_2)} = 0.$

Now recall that $\psi_{1,\varepsilon_m} = \varphi_{1,\varepsilon_m} - \tilde{\psi}_{1,\varepsilon_m} - (1 - \xi_{\varepsilon_m})\,\xi_1 \varphi^{-}_{\varepsilon_m}$ and that we know, by Lemma 13 and Lemma 10, that the last two terms converge to 0.

The following corollary can be obtained by the same method as in [AC93, p. 206] (see also [T02, p. 206]).

Corollary 15. *We can extract from $\{\psi_{1,\varepsilon_m}\}$ a subsequence which converges strongly in $L^2(M_1, g_1)$ and weakly in $H^1(M_1, g_1)$, and any such subsequence defines the limit form $\psi_1 \in H^1(M_1, g_1)$ such that*

$$\Delta_{M_1}\psi_1 = \lambda^p\,\psi_1 \quad \textit{and} \quad \|\psi_1\|_{L^2(M_1,g_1)} = 1.$$

Proof. We take a sequence $\{\psi_{1,\varepsilon_m}\}$ in $H^1(\Lambda^p M_1, g_1)$. From Proposition 11, this sequence is uniformly bounded in $H^1(\Lambda^p M_1, g_1)$. By the weak compactness theorem and the Rellich-Kondrachov theorem, there exist a subsequence $\{\psi_{1,\varepsilon_m}\}$ (we use the same notation for simplicity) and the limit $\psi_1 \in H^1(\Lambda^p M_1, g_1)$ such that $\psi_{1,\varepsilon_m} \to \psi_1$ weakly in $H^1(M_1, g_1)$ and strongly in $L^2(M_1, g_1)$ as $m \to \infty$.

Now, for any smooth p-form $\omega \in \Omega_0^p(M_1 - \{x_1\})$ with the support of ω in $(M_1 - \{x_1\})^\circ$, we have

$$\begin{aligned}
(\psi_1, \Delta_{g_1}\omega)_{L^2(M_1,g_1)} &= \lim_{m\to\infty}(\psi_{1,\varepsilon_m}, \Delta_{g_1}\omega)_{L^2(M_1)} \\
&= \lim_{m\to\infty}(\varphi_{\varepsilon_m}, \Delta_{\varepsilon_m}\omega)_{L^2(M_1(\varepsilon_m))} \\
&= \lim_{m\to\infty}(\lambda^p(M_{\varepsilon_m})\varphi_{\varepsilon_m}, \omega)_{L^2(M_{\varepsilon_m})} \\
&= \lambda^p \cdot \lim_{m\to\infty}(\varphi_{1,\varepsilon_m}, \omega)_{L^2(M_1(\varepsilon_m))} \\
&= \lambda^p \cdot \lim_{m\to\infty}(\psi_{1,\varepsilon_m}, \omega)_{L^2(M_1,g_1)} \\
&= \lambda^p \cdot (\psi_1, \omega)_{L^2(M_1,g_1)}.
\end{aligned}$$

Since $\Omega_0^p(M_1 - \{x_1\})$ is dense in $H^1(\Lambda^p M_1, g_1)$ by $\dim M_1 = n+1 \geq 2$, we obtain

$$\Delta_{g_1}\psi_1 = \lambda^p\,\psi_1 \qquad \text{weakly.}$$

Furthermore, by the regularity theorem of a weak solution to the strong elliptic equation, the limit form ψ_1 in fact is a smooth p-form on M_1.

Next, from the normalization $\|\varphi_{\varepsilon_m}\|_{L^2(M_{\varepsilon_m})} \equiv 1$ and Corollary 14, we obtain that the limit $\|\psi_1\|_{L^2(M_1)} = 1$. Hence, the limit ψ_1 is a non-zero smooth eigenform on (M_1, g_1) with the positive eigenvalue λ^p. □

Thus, we see that $\lambda^p = \lim_{m\to\infty}\lambda^p(M_{\varepsilon_m})$ belongs to the set of positive spectrum for p-forms on (M_1, g_1). Hence, we have finished the proof of Theorem B.

6. The proof of Theorem C

6.1. **The multiplicity of 0.** The dimension of the kernel of Δ_ε is given by the cohomology of M which can be calculated with the Mayer-Vietoris sequence associated to the covering $\{U_1, U_2\}$ introduced in Section 2; see Proposition 1 (recall that M is of dimension $n+1$).

If we remember that $H^p(M_j - B; \mathbb{R}) \cong H^p(M_j; \mathbb{R})$ for $p \leq n$, where B is a small ball, then we obtain that

$$H^p(M_\varepsilon; \mathbb{R}) \cong H^p(M_1; \mathbb{R}) \oplus H^p(M_2; \mathbb{R}) \text{ for } 1 \leq p \leq n,$$

while $H^p(M_\varepsilon; \mathbb{R}) \cong H^p(M_1; \mathbb{R}) \cong H^p(M_2; \mathbb{R}) \cong \mathbb{R}$ for $p = 0, n+1$.

The transplantation of the harmonic forms of M_1 in M_ε has been described in [AC93]. With the previous calculation, we have good candidates for a transplantation of the cohomology of M_2: for each $\psi_2 \in \mathrm{Ker}(\mathcal{D}_2)$ with $\|\psi_2\|_{L^2(M_2(1),g_2)} = 1$, we set

$$\tilde{\psi}_\varepsilon := (\tilde{\psi}_1, \psi_2) = U^*\big(\,\chi_1 P_\varepsilon(\sigma_2 \restriction_{\partial M_2(1)}),\, \sigma_2\big).$$

Now take $\varphi_\varepsilon \in \operatorname{Ker}(\Delta_\varepsilon)$. We apply φ_ε to the previous results, Proposition 11 and Remark 8. So, there exists a subsequence such that

$$\psi_{1,\varepsilon} \to \psi_1 \in \operatorname{Ker}(\Delta_1) \quad \text{ and } \quad \psi_{2,\varepsilon} \to \psi_2 \in \operatorname{Ker}(\mathcal{D}_2);$$

and only one of these two limits can be zero, that is, $\psi_1 = 0$ or $\psi_2 = 0$. The conclusion is that all the harmonic forms on M_ε can be approximated by forms like $\tilde{\psi}_\varepsilon$ or $\chi_\varepsilon \varphi_1$, with $\varphi_1 \in \operatorname{Ker}(\Delta_1)$.

As a consequence, we have

Corollary 16. *For $1 \le p \le n$, the two spaces $H^p(M_2;\mathbb{R})$ and $\operatorname{Ker}(\mathcal{D}_2)$ are isomorphic: $H^p(M_2;\mathbb{R}) \cong \operatorname{Ker}(\mathcal{D}_2)$.*

6.2. **The convergence of the positive spectrum.** The proof is made by induction. First we show that $\lim_{\varepsilon\to 0} \lambda_1^p(M_\varepsilon) = \lambda_1^p(M_1)$:

Proof. We know by Proposition A that $\limsup_{\varepsilon\to 0} \lambda_1^p(M_\varepsilon) \le \lambda_1^p(M_1)$ and by Theorem B that $\liminf_{\varepsilon\to 0} \lambda_1^p(M_\varepsilon)$ is in the positive spectrum of Δ_1, and as a consequence $\liminf_{\varepsilon\to 0} \lambda_1^p(M_\varepsilon) \ge \lambda_1^p(M_1)$. □

Now suppose that for all j with $1 \le j \le k$ one has $\lim_{\varepsilon\to 0} \lambda_j^p(M_\varepsilon) = \lambda_j^p(M_1)$. Then, we have to show that $\lim_{\varepsilon\to 0} \lambda_{k+1}^p(M_\varepsilon) = \lambda_{k+1}^p(M_1)$.

Proof. We know by Proposition A that $\limsup_{\varepsilon\to 0} \lambda_{k+1}^p(M_\varepsilon) \le \lambda_{k+1}^p(M_1)$.

Let $\{\varphi_\varepsilon^{(1)}, \dots, \varphi_\varepsilon^{(k+1)}\}$ be an orthonormal family of eigenforms on M_ε:

$$\Delta_\varepsilon \varphi_\varepsilon^{(j)} = \lambda_j^p(M_\varepsilon)\, \varphi_\varepsilon^{(j)},$$

and choose a sequence $\varepsilon_l \to 0$ (corresponding to $l \to \infty$) such that

$$\lim_{l\to\infty} \lambda_{k+1}^p(M_{\varepsilon_l}) = \liminf_{\varepsilon\to 0} \lambda_{k+1}^p(M_\varepsilon).$$

We apply each $\varphi_\varepsilon^{(j)}$ to the same decomposition as in Proposition 11. This gives a family $\{\psi_{1,\varepsilon}^{(1)}, \dots, \psi_{1,\varepsilon}^{(k+1)}\}$ bounded in $H^1(\Lambda^p M_1, g_1)$ and such that for each index j,

$$\lim_{\varepsilon\to 0} \|\varphi_{1,\varepsilon}^{(j)} - \psi_{1,\varepsilon}^{(j)}\|_{L^2(M_1(\varepsilon),g_1)} = 0,$$

while, as in Corollary 14,

$$\lim_{\varepsilon\to 0} \|\varphi_{2,\varepsilon}^{(j)}\|_{L^2(M_2(1),g_2)} = 0.$$

So, by extracting a subsequence, we can suppose that $\psi_{1,\varepsilon_l}^{(1)}, \dots, \psi_{1,\varepsilon_l}^{(k+1)}$ converge strongly in $L^2(M_1, g_1)$ and weakly in $H^1(M_1, g_1)$ and that the limit $\{\psi_1^{(1)}, \dots, \psi_1^{(k+1)}\}$ is orthonormal and satisfies for all j with $1 \le j \le k$,

$$\Delta_1 \psi_1^{(j)} = \lambda_j^p(M_1)\, \psi_1^{(j)} \quad \text{and} \quad \Delta_1 \psi_1^{(k+1)} = \liminf_{\varepsilon\to 0} \lambda_{k+1}^p(M_\varepsilon) \cdot \psi_1^{(k+1)}.$$

This shows that $\liminf_{\varepsilon\to 0} \lambda_{k+1}^p(M_\varepsilon) \ge \lambda_{k+1}^p(M_1)$ and completes the proof of Theorem C. □

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