Small eigenvalues of the Hodge-Laplacian with sectional curvature bounded below

Colette Anné and Junya Takahashi

29 May 2025

Dedicated to Professor Bruno Colbois on the occasion of his 65th birthday

Abstract

For each degree p and each natural number $k \geq 1$, we construct a one-parameter family of Riemannian metrics on any oriented closed manifold with volume one and the sectional curvature bounded below such that the k-th positive eigenvalue of the Hodge-Laplacian acting on differential p-forms converges to zero. This result imposes a constraint on the sectional curvature for our previous result in [AT24].

1 Introduction

We study the eigenvalue problems of the Hodge-Laplacian $\Delta = d\delta + \delta d$ acting on p-forms on a connected oriented closed Riemannian manifold (M^m, g) of dimension $m \geq 2$. The spectrum of the Hodge-Laplacian consists only of non-negative eigenvalues with finite multiplicity. We denote its **positive** eigenvalues counted with multiplicity by

$$\underbrace{0 = \cdots = 0}_{b_p(M)} < \lambda_1^{(p)}(M, g) \leq \lambda_2^{(p)}(M, g) \leq \cdots \leq \lambda_k^{(p)}(M, g) \leq \cdots ,$$

where the multiplicity of the eigenvalue 0 is equal to the p-th Betti number $b_p(M)$ of M, by the Hodge-Kodaira-de Rham theory. In particular, it is independent of a choice of Riemannian metrics.

In our previous paper [AT24, Theorem 1.2], for any fixed degree p with $1 \leq p \leq m-1$, we constructed a one-parameter family of Riemannian metrics $\{\overline{g}_{p,L}\}_{L>1}$ on

2020 *Mathematics Subject Classification. Primary* $58J50$*; Secondary* $35P15$, $53C21$, $58C40$.
Key Words and Phrases. Hodge-Laplacian, differential forms, eigenvalues, surgery, sectional curvature.

a connected oriented closed m-dimensional manifold M with volume one such that for any natural number $k \geq 1$

$$\lambda_k^{(p)}(M, \overline{g}_{p,L}) \longrightarrow 0 \text{ as } L \longrightarrow \infty. \tag{1.1}$$

If M is the m-dimensional standard sphere $\mathbb{S}^m$, then we can choose such a family of Riemannian metrics to have non-negative sectional curvature ([AT24, Theorem 1.1]). These metrics are also positive Ricci curvature for $m \geq 4$. But, for $m = 3$ and $p = 1$, they are flat on some domain.

For a general closed manifold M, however, the same result cannot hold any longer. In fact, there exist some topological obstructions to admit a Riemannian metric on M with non-negative Ricci curvature. One of the most famous obstructions is the Bochner theorem: If a closed manifold M admits a Riemannian metric with non-negative Ricci curvature, then the first Betti number must hold $b_1(M) \leq b_1(T^m) = m$.

Because of such a topological obstruction, we weaken a curvature constraint of a general closed manifold M from non-negative sectional curvature to the sectional curvature bounded below by a negative constant.

In the present paper, for any closed manifold M of dimension $m \geq 2$, we construct such a family of Riemannian metrics with the sectional curvature uniformly bounded below.

Theorem 1.1. *Let M^m be a connected oriented closed manifold of dimension $m \geq 2$. For a given degree p with $0 \leq p \leq m$, a natural number k and any $\varepsilon > 0$, there exists a one-parameter family of Riemannian metrics $\{\overline{g}_{\varepsilon,p,k}\}_{\varepsilon>0}$ on M with volume one and the sectional curvature uniformly bounded below $K_{\overline{g}_{\varepsilon,p,k}} \geq -\kappa$ for some constant $\kappa > 0$ such that*

$$\lambda_k^{(p)}(M, \overline{g}_{\varepsilon,p,k}) \longrightarrow 0 \quad \text{as } \varepsilon \longrightarrow 0.$$

The construction of this one-parameter family of Riemannian metrics is as follows: We take an embedded p-dimensional sphere $\mathbb{S}^p$ into M whose normal bundle is trivial. Then, in a tubular neighborhood of $\mathbb{S}^p$, we change a disk of the normal direction to get longer and thinner, while keeping its sectional curvature uniformly bounded below.

Remark 1.2. *(i) The Riemannian metrics $\overline{g}_{\varepsilon,p,k}$ in Theorem 1.1 depend on the degree p and the number k of the positive eigenvalues.*

(ii) For the Riemannian metrics $\overline{g}_{\varepsilon,p,k}$ on M in Theorem 1.1, from the proof, we find that the diameter $\mathrm{diam}(M, \overline{g}_{\varepsilon,p,k}) \longrightarrow \infty$ as $\varepsilon \longrightarrow 0$.

(iii) For the rough Laplacian $\overline{\Delta} = \nabla^*\nabla$ acting on p-forms and tensor fields of any type, the same statement also holds (See Remark 5.3 (ii)).

The Riemannian metrics $\overline{g}_{\varepsilon,p,k}$ in Theorem 1.1 depend also on the degree p of differential forms. However, by taking $m-1$ distinct embedded spheres $\mathbb{S}^0, \mathbb{S}^1, \mathbb{S}^2, \ldots, \mathbb{S}^{m-2}$ in M (see Lemma 5.1) and applying the same construction in Theorem 1.1 to each sphere, we can obtain a family of Riemannian metrics $\overline{g}_{\varepsilon,k}$ on M, which are independent of all the degrees $p = 0, 1, \ldots, m$, with small eigenvalues for all the degrees $p = 0, 1, 2, \ldots, m$.

Theorem 1.3. *Let M^m be a connected oriented closed manifold of dimension $m \geq 2$. For any $\varepsilon > 0$ and a natural number k, there exists a one-parameter family of Riemannian metrics $\{\overline{g}_{\varepsilon,k}\}_{\varepsilon>0}$ on M with volume one and the sectional curvature uniformly bounded below $K_{\overline{g}_{\varepsilon,k}} \geq -\kappa$ for some constant $\kappa > 0$ such that for any degree p with $0 \leq p \leq m$*

$$\lambda_k^{(p)}(M, \overline{g}_{\varepsilon,k}) \longrightarrow 0 \quad \textit{as } \varepsilon \longrightarrow 0.$$

Remark 1.4. *As a consequence of Theorem 1.3, we find that there exists no positive lower bound for the positive eigenvalue of the Hodge-Laplacian on p-forms for any degree p with $1 \leq p \leq m-1$ depending only on the dimension, the volume and a lower bound of the sectional curvature.*

From Remark 1.2 (*ii*), it is a natural question to ask the case where the diameter is bounded in addition. In this case, it would be expected to exist a positive lower bound for the positive eigenvalues of the Hodge-Laplacian for all the degree $p = 0, 1, \ldots, m$. This was conjectured by J. Lott [Lo04, p.918] (See Conjecture 6.2).

The present paper is organized as follows: In Section 2, we fix notations and recall basic properties of the Hodge-Laplacian. In Section 3, we consider the hyperbolic dumbbell and a connected sum of its k copies. In Section 4, we construct a family of Riemannian metrics on any closed manifold M, and in Section 5, we prove that such Riemannian manifolds have small eigenvalues, which completes the proof of Theorem 1.1. In Section 6, we discuss some remarks and further studies.

Acknowledgement. The authors would like to thank the referees for helpful comments. The second named author was partially supported by the Grants-in-Aid for Scientific Research (C), Japan Society for the Promotion of Science, No. 16K05117.

2 Notations and basic facts

We fix the notations used in the present paper. We use the same notations as in [AT24]. Let (M^m, g) be a connected oriented closed Riemannian manifold of dimension $m \geq 2$. The metric g defines the volume element $d\mu_g$ and the scalar product on the fibers of any tensor bundle. The L^2-inner product on the space of all smooth p-forms $\Omega^p(M)$ is defined as, for any p-forms φ, ψ on M

$$(\varphi, \psi)_{L^2(M,g)} := \int_M \langle \varphi, \psi \rangle d\mu_g \quad \text{and} \quad \| \varphi \|^2_{L^2(M,g)} := (\varphi, \varphi)_{L^2(M,g)}.$$

The space of L^2 p-forms $L^2(\Lambda^p M, g)$ is the completion of $\Omega^p(M)$ with respect to this L^2-norm.

We now recall the basic properties used in the present paper:

Lemma 2.1. (1) *The Hodge duality: For all* $p = 0, 1, \ldots, m$ *and any* $k \geq 1$*, since* $\Delta * = * \Delta$*, we have*

$$\lambda_k^{(m-p)}(M, g) = \lambda_k^{(p)}(M, g).$$

(2) *The scaling change of metrics: For a positive constant* $a > 0$ *and for all* $p = 0, 1, \ldots, m$ *and any* $k \geq 1$*, we have*

$$\lambda_k^{(p)}(M, ag) = a^{-1}\, \lambda_k^{(p)}(M, g).$$

(3) *The normalization of the volume: If we set the new Riemannian metric*

$$\overline{g} := \mathrm{vol}(M, g)^{-\frac{2}{m}}\, g, \tag{2.1}$$

then we have $\mathrm{vol}(M, \overline{g}) \equiv 1$.

In particular, from the properties (2) and (3), we have

$$\lambda_k^{(p)}(M, \overline{g}) = \mathrm{vol}(M, g)^{\frac{2}{m}}\, \lambda_k^{(p)}(M, g) \tag{2.2}$$

for any p and k.

3 The hyperbolic dumbbell and its connected sum

3.1 The hyperbolic dumbbell

Following Boulanger and Courtois [BC22], Section 5, pp.3626–3628, we recall the n-dimensional hyperbolic dumbbell $(C_\varepsilon, g_\varepsilon)$ with parameter $\varepsilon > 0$.

For any $\varepsilon > 0$, we first consider the n-dimensional hyperbolic cylinder $C_{0,\varepsilon} := [-L, L] \times \mathbb{S}^{n-1}$ with the Riemannian metric

$$g_\varepsilon = dr \oplus \varepsilon^2 \cosh^2(r) g_{\mathbb{S}^{n-1}} \quad (\varepsilon > 0) \tag{3.1}$$

for $-L \leq r \leq L$, where $L := |\log \varepsilon|$ ($\varepsilon = e^{-L}$) for short and $g_{\mathbb{S}^{n-1}}$ denotes the standard Riemannian metric on the $n-1$ dimensional standard sphere $\mathbb{S}^{n-1}$ of constant curvature one.

Let B_1, B_2 be two n-dimensional spheres with the standard metrics from which n-dimensional disks are removed. We glue B_1, B_2 to the boundary of this hyperbolic cylinder $C_{0,\varepsilon}$, identifying ∂B_1 with the left-side boundary $\{-L\} \times \mathbb{S}^{n-1}$ and ∂B_2 with the right-side boundary $\{L\} \times \mathbb{S}^{n-1}$. It means that the removed disks on B_1, B_2 have the radius $\varepsilon \cosh(|\log \varepsilon|) \to 1/2$ as $\varepsilon \to 0$. The resulting manifold is diffeomorphic to $\mathbb{S}^n$. We extend the Riemannian metric g_ε on the hyperbolic cylinder $C_{0,\varepsilon}$ to the whole Riemannian metric on $\mathbb{S}^n$ which is independent of ε on the both-sides B_1, B_2. In addition, we can choose the extended Riemannian metric as the standard sphere metrics on the both-sides B_1, B_2 away from their boundaries. We also denote by g_ε this extended Riemannian metric, and we call the resulting Riemannian manifold the n-dimensional hyperbolic dumbbell denoted by $(C_\varepsilon, g_\varepsilon)$ (see Figure 1) .

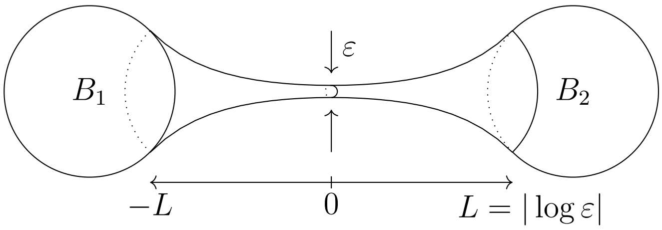

Figure 1: the hyperbolic dumbbell $(C_\varepsilon, g_\varepsilon)$

We precisely exhibit the way of connecting of g_ε and the standard metric of the sphere as follows: From the symmetry of the hyperbolic cylinder $C_{0,\varepsilon}$, it is enough to consider the connecting part corresponding to $r = L = |\log \varepsilon|$.

We introduce the new coordinate $s := r - L = r + \log \varepsilon$, then

$$f_\varepsilon(s) := \varepsilon \cosh(s + L) = \frac{1}{2}e^s + \frac{\varepsilon^2}{2}e^{-s} \tag{3.2}$$

is the warping function of g_ε on $-2L \leq s \leq 0$. We set

$$h(s) := \sin\left(s + \frac{\pi}{6}\right) \quad (0 \leq s \leq \frac{\pi}{12}). \tag{3.3}$$

To connect these two positive functions $f_\varepsilon(s)$ and $h(s)$ smoothly, we define the new function $F_\varepsilon(s)$ as follows:

$$F_\varepsilon(s) := \chi(s) f_\varepsilon(s) + \big(1 - \chi(s)\big) h(s) \quad (0 \leq s \leq \frac{\pi}{12}), \tag{3.4}$$

where $\chi(s)$ is a smooth cut-off function satisfying

$$\chi(s) = \begin{cases} 1 & (0 \leq s \leq \frac{\pi}{36}), \\ 0 & (\frac{\pi}{18} \leq s \leq \frac{\pi}{12}). \end{cases}$$

By (3.2) and (3.2), the equation (3.4) is written as

$$F_\varepsilon(s) = \left\{ \chi(s)\frac{1}{2}e^s + \big(1 - \chi(s)\big)h(s) \right\} + \frac{\varepsilon^2}{2}e^{-s}\chi(s).$$

If we take ε small enough, the term $\frac{\varepsilon^2}{2}e^{-s}\chi(s)$ and its derivatives are also small enough. Hence, there exists an $\varepsilon_0 > 0$ such that for all $0 < \varepsilon < \varepsilon_0$, $F_\varepsilon(s)$, $F'_\varepsilon(s)$ and

$F''_\varepsilon(s)$ are uniformly bounded on $0 \le s \le \frac{\pi}{12}$. In particular, since $f_\varepsilon(s)$ and $h(s)$ are monotone increasing, we see

$$\begin{aligned} 0.5 < f_\varepsilon(0) \le f_\varepsilon(s) \le f_\varepsilon(\tfrac{\pi}{12}) < e^{\pi/12} < 0.7, \\ 0.5 = \sin(\tfrac{\pi}{6}) \le h(s) \le \sin(\tfrac{\pi}{4}) = \tfrac{1}{\sqrt{2}} < 0.8. \end{aligned}$$

Thus, we have

$$0.5 \le F_\varepsilon(s) < 0.8 \quad (0 \le s \le \tfrac{\pi}{12}). \tag{3.5}$$

Now, if we take a Riemannian metric around the connecting part as

$$ds^2 \oplus F_\varepsilon^2(s)\, g_{\mathbb{S}^{n-1}} \quad (0 \le s \le \tfrac{\pi}{12}), \tag{3.6}$$

then the whole Riemannian metric g_ε on the hyperbolic dumbbell C_ε is smooth, and coincides with the Riemannian metric on the hyperbolic cylinder $C_{0,\varepsilon}$ and coincides with the standard sphere metric on B_2. In fact, since $F_\varepsilon(s) = f_\varepsilon(s) = \varepsilon\cosh(s)$ for $0 \le s \le \frac{\pi}{36}$, we have

$$ds^2 \oplus F_\varepsilon^2(s)\, g_{\mathbb{S}^{n-1}} = ds^2 \oplus \varepsilon^2\cosh^2(s)\, g_{\mathbb{S}^{n-1}} \quad \text{on } [0, \tfrac{\pi}{36}] \times \mathbb{S}^{n-1},$$

which coincides with the Riemannian metric on the hyperbolic cylinder. Since $F_\varepsilon(s) = h(s) = \sin(s + \frac{\pi}{6})$ for $\frac{\pi}{18} \le s \le \frac{\pi}{12}$, we have

$$ds^2 \oplus F_\varepsilon^2(s)\, g_{\mathbb{S}^{n-1}} = ds^2 \oplus \sin^2(s + \tfrac{\pi}{6})\, g_{\mathbb{S}^{n-1}} \quad \text{on } [\tfrac{\pi}{18}, \tfrac{\pi}{12}] \times \mathbb{S}^{n-1},$$

which coincides with the standard sphere metric on B_2.

Lemma 3.1 (sectional curvature of warped product manifolds)**.** *For a Riemannian manifold (N, h) and a smooth positive function $f(r)$ on the interval I, we consider the warped product manifold $(M, g_f) := (I \times N, dr^2 \oplus f^2(r)h)$. For orthonormal vectors X and Y on (N, h), the vectors $\widetilde{X} := f(r)^{-1}X$, $\widetilde{Y} := f(r)^{-1}Y$ on M are orthonormal and perpendicular to $\partial_r = \frac{\partial}{\partial r}$ with respect to the metric g_f.*

Then, the sectional curvatures K_M of (M, g_f) are given as follows:

(i) $K_M(\partial_r, \widetilde{X}) = -\dfrac{f''(r)}{f(r)}$,

(ii) $K_M(\widetilde{X}, \widetilde{Y}) = \dfrac{K_N(X, Y) - (f'(r))^2}{f^2(r)}$.

In particular, if $(N^n, h) = (\mathbb{S}^n, g_{\mathbb{S}^n})$, then $K_N(X, Y) \equiv 1$.

For the proof of this lemma, see Petersen [Pet16], **4.2.3**, p.121.

Lemma 3.2. *The sectional curvature K_{C_ε} on the hyperbolic dumbbell $(C_\varepsilon, g_\varepsilon)$ is uniformly bounded below in ε. That is, there exists some positive constant $\kappa' > 0$ independent of ε such that*

$$K_{C_\varepsilon} \ge -\kappa'.$$

Proof. Since the metric on the both-sides bumps is independent of ε, we have only to show the boundedness on the central part of C_ε.

We use the same notation as in Lemma 3.1. On the hyperbolic cylinder $C_{0,\varepsilon}$, from (3.1), we have

$$K(\partial_r, \widetilde{X}) = -\frac{\varepsilon \cosh''(r)}{\varepsilon \cosh(r)} = -\frac{\cosh(r)}{\cosh(r)} = -1,$$

$$K(\widetilde{X}, \widetilde{Y}) = \frac{1 - \big(\varepsilon \cosh'(r)\big)^2}{\big(\varepsilon \cosh(r)\big)^2} \geq -\frac{\varepsilon^2 \sinh^2(r)}{\varepsilon^2 \cosh^2(r)} = -\frac{\sinh^2(r)}{\cosh^2(r)} \geq -1.$$

Thus, the sectional curvature on the hyperbolic cylinder $C_{0,\varepsilon}$ is uniformly bounded below by -1.

Next, around the right-side of the boundary $\partial C_{0,\varepsilon}$, since the Riemannian metric is expressed as (3.6), there exist positive constants $\kappa_1, \kappa_2 > 0$ independent of ε such that

$$K(\partial_r, \widetilde{X}) = -\frac{F''_\varepsilon(s)}{F_\varepsilon(s)} \geq -\kappa_1,$$

$$K(\widetilde{X}, \widetilde{Y}) = \frac{1 - \big(F'_\varepsilon(s)\big)^2}{F^2_\varepsilon(s)} \geq -\kappa_2.$$

Therefore, we find that the sectional curvature on the hyperbolic dumbbell $(C_\varepsilon, g_\varepsilon)$ is uniformly bounded below in ε. □

Furthermore, the volume of the hyperbolic dumbbell $(C_\varepsilon, g_\varepsilon)$ is uniformly bounded in ε.

Lemma 3.3. *There exist two positive constants $V_1, V_2 > 0$ independent of ε such that*

$$0 < V_1 \leq \mathrm{vol}(C_\varepsilon, g_\varepsilon) \leq V_2. \tag{3.7}$$

Proof. We first estimate the volume of the n-dimensional hyperbolic cylinder $(C_{0,\varepsilon}, g_\varepsilon)$ from above. The volume of $(C_{0,\varepsilon}, g_\varepsilon)$ is

$$\begin{aligned}\mathrm{vol}(C_{0,\varepsilon}, g_\varepsilon) &= 2\int_0^L \int_{\mathbb{S}^{n-1}} \big(\varepsilon \cosh(r)\big)^{n-1} dr d\mu_{g_{\mathbb{S}^{n-1}}} \\ &= 2\,\mathrm{vol}(\mathbb{S}^{n-1})\,\varepsilon^{n-1} \int_0^L \cosh^{n-1}(r) dr.\end{aligned}$$

Since $\cosh(r) \leq e^r$ for $r \geq 0$ and $L = -\log \varepsilon$, we have

$$\begin{aligned}\int_0^L \cosh^{n-1}(r) dr &\leq \int_0^L e^{(n-1)r} dr = \frac{1}{n-1}(e^{(n-1)L} - 1) \\ &\leq \frac{1}{n-1} e^{(n-1)L} = \frac{1}{n-1}\varepsilon^{-(n-1)},\end{aligned} \tag{3.8}$$

and finally

$$\mathrm{vol}(C_{0,\varepsilon}, g_\varepsilon) \le \frac{2}{n-1}\,\mathrm{vol}(\mathbb{S}^{n-1}).$$

Thus, the volume of $(C_{0,\varepsilon}, g_\varepsilon)$ is finite for ε. Since the volumes of B_1 and B_2 are bounded above in ε, the total volume of the hyperbolic dumbbell $(C_\varepsilon, g_\varepsilon)$ is unform bounded above in ε.

On the other hand, since the metrics on B_1 and B_2 away from their boundaries coincide with the standard sphere metrics, there exists a uniform lower bound of the volume:

$$\mathrm{vol}(C_\varepsilon, g_\varepsilon) \ge \mathrm{vol}(B_1) + \mathrm{vol}(B_2) \ge \frac{1}{2}\mathrm{vol}(\mathbb{S}^n) + \frac{1}{2}\mathrm{vol}(\mathbb{S}^n) = \mathrm{vol}(\mathbb{S}^n) := V_1.$$

□

3.2 The connected sum of k-copies of the hyperbolic dumbbell

Next, we perform the connected sum of k-copies of the hyperbolic dumbbell in series. The resulting Riemannian manifold is denoted by $C_{k,\varepsilon} = \overset{k}{\sharp} C_\varepsilon$ with the periodic metric $g_{C_{k,\varepsilon}}$ (see the Figure 2).

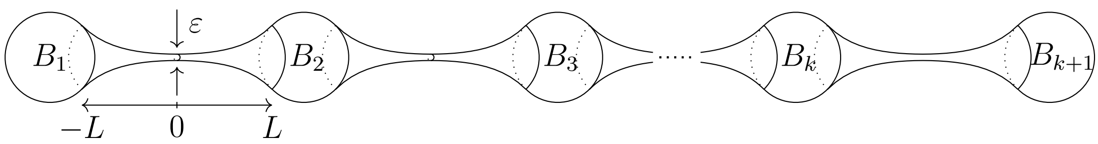

Figure 2: the connected sum of k-copies of the hyperbolic dumbbell

From the construction, $(C_{k,\varepsilon}, g_{C_{k,\varepsilon}})$ also satisfies the following property:

Lemma 3.4. *(i) The sectional curvature of $(C_{k,\varepsilon}, g_{C_{k,\varepsilon}})$ is uniformly bounded below in ε;*

(ii) The volume of $(C_{k,\varepsilon}, g_{C_{k,\varepsilon}})$ is uniformly bounded in ε.

4 Construction of the Riemannian metrics

We construct a one-parameter family of Riemannian metrics $\{\overline{g}_\varepsilon\}_{\varepsilon>0}$ on any closed manifold M with the volume one and the sectional curvature uniformly bounded below: $K_{\overline{g}_\varepsilon} \ge -\kappa$, where $\kappa > 0$ is independent of ε.

Let M be a connected oriented closed C^∞-manifold of dimension $m \ge 2$. For a given degree p with $0 \le p \le m-2$, we can take an embedded p-dimensional sphere

$\mathbb{S}^p$ into M whose normal bundle is trivial. Then, a closed tubular neighborhood $\mathrm{Tub}(\mathbb{S}^p)$ of $\mathbb{S}^p$ in M can be identified with

$$\mathrm{Tub}(\mathbb{S}^p) \cong \mathbb{S}^p \times \mathbb{D}^{m-p}, \tag{4.1}$$

where $\mathbb{D}^n$ denotes the n-dimensional closed unit disk in $\mathbb{R}^n$.

We now take any Riemannian metric $g_{p,M}$ on M such that $g_{p,M}$ on $\mathrm{Tub}(\mathbb{S}^p)$ is the product metric of $g_{\mathbb{S}^p}$ on $\mathbb{S}^p$ and the standard Euclidean metric $g_{\mathbb{R}^{m-p}}$ on $\mathbb{D}^{m-p}$:

$$g_{p,M} = g_{\mathbb{S}^p} \oplus g_{\mathbb{R}^{m-p}} \quad \text{on } \mathrm{Tub}(\mathbb{S}^p) = \mathbb{S}^p \times \mathbb{D}^{m-p}. \tag{4.2}$$

We decompose M into the two components H_1, H_2:

$$\begin{aligned} H_1 &:= \mathbb{S}^p \times \mathbb{D}^{m-p}, \\ H_2 &:= \overline{M \setminus H_1} = \overline{M \setminus (\mathbb{S}^p \times \mathbb{D}^{m-p})}. \end{aligned} \tag{4.3}$$

Then, while fixing the metric $g_{p,M}$ on H_2, we change the metric $g_{p,M}$ on H_1 to a new metric.

For any real number $\varepsilon > 0$ and any natural number $k \geq 1$, as constructed in the previous sub-section **3.2**, we take the connected sum of k-copies of the $(m-p)$-dimensional hyperbolic dumbbell $C_{k,\varepsilon} = \overset{k}{\sharp} C_\varepsilon$ with the Riemannian metrics $g_{C_{k,\varepsilon}}$, and glue it to the second factor $\mathbb{D}^{m-p}$ of H_1 (See the Figure 3 below). This gluing can be done independently of ε. We also use the same notation of this new Riemannian metrics $g_{C_{k,\varepsilon}}$ on the gluing $C_{k,\varepsilon} \sharp \, \mathbb{D}^{m-p} \cong \mathbb{D}^{m-p}$.

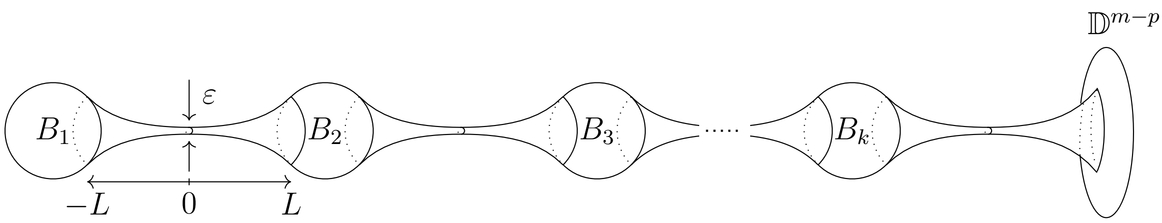

Figure 3: gluing $C_{k,\varepsilon}$ to $\mathbb{D}^{m-p}$

Thus, we obtain a one-parameter family of Riemannian metrics $g_{\varepsilon,p,k}$ on $H_1 = \mathrm{Tub}(\mathbb{S}^p) = \mathbb{S}^p \times \mathbb{D}_\varepsilon^{m-p}$ as

$$g_{\varepsilon,p,k} := g_{\mathbb{S}^p} \oplus g_{C_{k,\varepsilon}}. \tag{4.4}$$

Then, we define the one-parameter family of Riemannian metrics $\{g_{\varepsilon,p,k}\}_{\varepsilon>0}$ on M as

$$g_{\varepsilon,p,k} := \begin{cases} g_{\mathbb{S}^p} \oplus g_{C_{k,\varepsilon}} & \text{on } H_1 = \mathbb{S}^p \times \mathbb{D}_\varepsilon^{m-p}, \\ g_{p,M} & \text{on } H_2 = \overline{M \setminus H_1}. \end{cases} \tag{4.5}$$

Finally, we normalize this metric $g_{\varepsilon,p,k}$ whose total volume is one. That is, we define

$$\overline{g}_{\varepsilon,p,k} := \mathrm{vol}(M, g_{\varepsilon,p,k})^{-\frac{2}{m}} \, g_{\varepsilon,p,k} \quad \text{on } M. \tag{4.6}$$

Then, $\mathrm{vol}(M, \overline{g}_{\varepsilon,p,k}) \equiv 1$.

Lemma 4.1 (Estimates of the volume)**.** *There exist positive constants $V, A, B > 0$ independent of ε and k such that*

$$0 < V \leq \mathrm{vol}(M, g_{\varepsilon,p,k}) \leq Ak + B. \tag{4.7}$$

Proof. For an upper bound, from Lemma 3.3, we have

$$\begin{aligned}
\mathrm{vol}(M, g_{\varepsilon,p,k}) &= \mathrm{vol}(H_1, g_{\varepsilon,p,k}) + \mathrm{vol}(H_2, g_{\varepsilon,p,k}) \\
&= \mathrm{vol}(\mathbb{S}^p) \cdot \mathrm{vol}(C_{k,\varepsilon}, g_{C_{k,\varepsilon}}) + \mathrm{vol}(H_2, g_{p,M}) \\
&\leq \mathrm{vol}(\mathbb{S}^p) \cdot \mathrm{vol}(C_{1,\varepsilon}, g_{C_{1,\varepsilon}})\, k + \mathrm{vol}(H_2, g_{p,M}) \\
&\leq Ak + B,
\end{aligned}$$

where $A, B > 0$ are some constants independent of ε and k.

For a positive lower bound, from Lemma 3.3, we also have

$$\begin{aligned}
\mathrm{vol}(M, g_{\varepsilon,p,k}) &= \mathrm{vol}(H_1, g_{\varepsilon,p,k}) + \mathrm{vol}(H_2, g_{\varepsilon,p,k}) \\
&\geq \mathrm{vol}(H_1, g_{\varepsilon,p,k}) = \mathrm{vol}(\mathbb{S}^p) \cdot \mathrm{vol}(C_{k,\varepsilon}, g_{C_{k,\varepsilon}}) \\
&\geq \mathrm{vol}(\mathbb{S}^p) \cdot \mathrm{vol}(C_{1,\varepsilon}, g_{C_{1,\varepsilon}}) \geq \mathrm{vol}(\mathbb{S}^p) \cdot V_1 > 0.
\end{aligned}$$

□

Hence, from the property of the sectional curvature

$$K_{(M,\overline{g}_{\varepsilon,p,k})} = \mathrm{vol}(M, g_{\varepsilon,p,k})^{\frac{2}{m}}\, K_{(M,g_{\varepsilon,p,k})}$$

and Lemma 4.1, we find that the family of volume-normalized Riemannian metrics $\{\overline{g}_{\varepsilon,p,k}\}_{\varepsilon>0}$ on M defined in (4.6) satisfies the same properties as in Lemma 3.4.

Lemma 4.2. *(i) The sectional curvature of $(M, \overline{g}_{\varepsilon,p,k})$ is uniformly bounded below in ε;*

(ii) The volume of $(M, \overline{g}_{\varepsilon,p,k})$ is identically one.

5 The proof of Theorem 1.1

We give the proof of Theorem 1.1 by using the min-max principle for the Hodge-Laplacian acting on co-exact forms. We denote by $\lambda_k'^{(p)}(M, g)$ and $\lambda_k''^{(p)}(M, g)$ the k-th eigenvalues of the Hodge-Laplacian acting on exact and co-exact forms, respectively, which are always positive. Theorem 1.1 is a corollary of Lemma 5.1.

Lemma 5.1 (Small eigenvalues)**.** *Let p be an integer with $0 \leq p \leq m-2$. For any integer $k \geq 1$ and any real number $\varepsilon > 0$, there exists a positive constant $C(m, p, \overline{k}) > 0$ independent of ε such that*

$$\lambda_k''^{(p)}(M, \overline{g}_{\varepsilon,p,\overline{k}}) \leq \frac{C(m, p, \overline{k})}{|\log \varepsilon|^2},$$

where $\overline{k} := k + b_p(M)$, where $b_p(M)$ is the p-th Betti number of M.

Remark 5.2. *In the case of $p = m-1, m$, by the Hodge duality $\lambda_k^{\prime\prime(p)} = \lambda_k^{\prime(m-p)}$, we can reduce to the case of $p = 1, 0$, respectively. Therefore, the same statement as Lemma 5.1 for exact p-forms still holds in the case of $p = m-1, m$.*

Proof. To prove the estimate in Lemma 5.1, we use the min-max principle for the Hodge-Laplacian acting on co-exact p-forms. Since the space of co-closed p-forms modulo co-exact p-forms is that of harmonic p-forms, whose dimension is $b_p(M)$, we may construct $\overline{k} = k + b_p(M)$ test co-closed p-forms φ_i on M.

Let χ_i be the linear cut-off functions on $C_{\overline{k},\varepsilon}$ as follows (See Figure 4): For χ_1, we define χ_1 as

$$\chi_1 := \begin{cases} 1 & \text{for } r \le -L, \\ -\dfrac{r}{L} & \text{for } -L \le r \le 0, \\ 0 & \text{for } 0 \le r. \end{cases}$$

For χ_i $(i = 2, 3, \ldots, \overline{k})$, we define χ_i periodically as

$$\chi_i := \begin{cases} 0 \quad \text{for } r \le (2i-4)L + (i-2)\frac{2}{3}\pi, \\ \dfrac{1}{L}\left(r - (2i-4)L - (i-2)\frac{2}{3}\pi\right) \\ \qquad \text{for } (2i-4)L + (i-2)\frac{2}{3}\pi \le r \le (2i-3)L + (i-2)\frac{2}{3}\pi, \\ 1 \quad \text{for } (2i-3)L + (i-2)\frac{2}{3}\pi \le r \le (2i-3)L + (i-1)\frac{2}{3}\pi, \\ -\dfrac{1}{L}\left(r - (2i-2)L - (i-1)\frac{2}{3}\pi\right) \\ \qquad \text{for } (2i-3)L + (i-1)\frac{2}{3}\pi \le r \le (2i-2)L + (i-1)\frac{2}{3}\pi, \\ 0 \quad \text{for } (2i-2)L + (i-1)\frac{2}{3}\pi \le r. \end{cases}$$

From the construction, the interiors of the supports $\mathrm{supp}(\chi_i)^\circ$ are mutually disjoint:

$$\mathrm{supp}(\chi_i)^\circ \cap \mathrm{supp}(\chi_j)^\circ = \emptyset \quad (i \ne j).$$

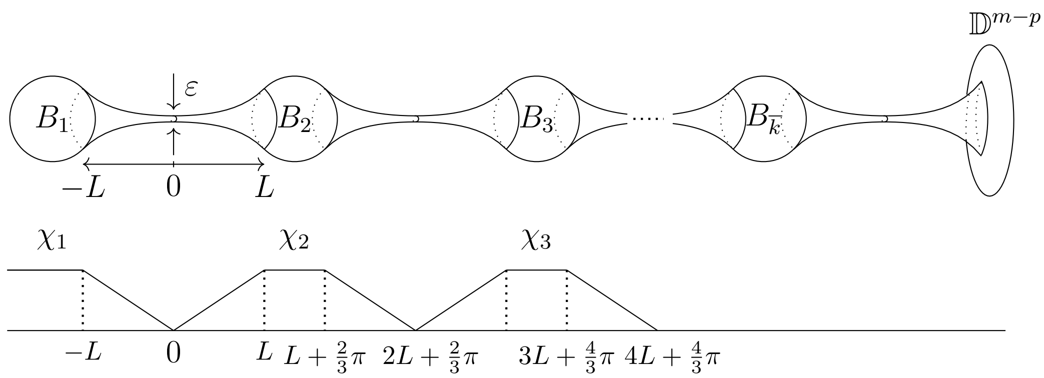

Figure 4: the cut-off functions χ_i on $C_{\overline{k},\varepsilon}$

Then, we take the test p-forms φ_i as follows:

$$\varphi_i := \begin{cases} \chi_i(r)\, v_{\mathbb{S}^p} & \text{on } H_1 = \mathbb{S}^p \times \mathbb{D}^{m-p}, \\ 0 & \text{on } H_2 = \overline{M \setminus H_1}, \end{cases} \tag{5.1}$$

where $v_{\mathbb{S}^p}$ is the volume form of $\mathbb{S}^p$.

These φ_i are co-closed p-form on $(M, g_{\varepsilon,p,\overline{k}})$. In fact, since the metric is product on H_1, we have on H_1

$$\begin{aligned} \delta_{g_{\varepsilon,p,\overline{k}}} \varphi_i &= \delta_{g_{\varepsilon,p,\overline{k}}}\big(\chi_i\, v_{\mathbb{S}^p}\big) = \big(\delta_{g_{\varepsilon,p,\overline{k}}}\chi_i\big)\, v_{\mathbb{S}^p} + \chi_i\, \big(\delta_{g_{\varepsilon,p,\overline{k}}} v_{\mathbb{S}^p}\big) \\ &= 0 + \chi_i\, \big(\delta_{g_{\mathbb{S}^p}} v_{\mathbb{S}^p}\big) \equiv 0. \end{aligned}$$

Since the supports of the family $\{\varphi_i\}_{i=1}^{\overline{k}}$ are disjoint up to measure 0, the min-max principle for the Hodge-Laplacian on co-exact forms gives us

$$\begin{aligned} \lambda_k''^{(p)}(M, g_{\varepsilon,p,\overline{k}}) &\le \max_{i=1,2,\ldots,\overline{k}} \left\{ \frac{\|d\,\varphi_i\|^2_{L^2(M, g_{\varepsilon,p,\overline{k}})}}{\|\,\varphi_i\,\|^2_{L^2(M, g_{\varepsilon,p,\overline{k}})}} \right\} \\ &= \max_{i=1,2,\ldots,\overline{k}} \left\{ \frac{\|d\,\varphi_i\|^2_{L^2(H_1, g_{\varepsilon,p,\overline{k}})}}{\|\,\varphi_i\,\|^2_{L^2(H_1, g_{\varepsilon,p,\overline{k}})}} \right\}, \end{aligned} \tag{5.2}$$

because of $\varphi_i \equiv 0$ on H_2.

First, we estimate the numerator of (5.2) from above. Since

$$d\,\varphi_i = d\big(\chi_i(r)\, v_{\mathbb{S}^p}\big) = \chi'(r) dr \wedge v_{\mathbb{S}^p} \quad \text{on } H_1 \tag{5.3}$$

and $|\chi'(r)|^2 = \dfrac{1}{L^2}$ on $[-L, 0]$, we have

$$\begin{aligned} \|d\,\varphi_i\|^2_{L^2(H_1, g_{\varepsilon,p,\overline{k}})} &= \int_{\mathbb{S}^p} \int_{C_{\varepsilon,1}} \big|\chi'(r) dr \wedge v_{\mathbb{S}^p}\big|^2_{g_{\varepsilon,p,\overline{k}}} (\varepsilon \cosh(r))^{m-p-1}\, dr d\mu_{\mathbb{S}^{m-p-1}} d\mu_{\mathbb{S}^p} \\ &= \varepsilon^{m-p-1} \int_{-L}^{0} |\chi'(r)|^2 \cosh^{m-p-1}(r) dr \int_{\mathbb{S}^{m-p-1}} d\mu_{\mathbb{S}^{m-p-1}} \int_{\mathbb{S}^p} d\mu_{\mathbb{S}^p} \\ &= \varepsilon^{m-p-1} \int_{-L}^{0} \frac{1}{L^2} \cosh^{m-p-1}(r) dr \cdot \mathrm{vol}(\mathbb{S}^{m-p-1})\, \mathrm{vol}(\mathbb{S}^p) \\ &= \frac{\varepsilon^{m-p-1}}{L^2} \int_0^L \cosh^{m-p-1}(r) dr \cdot \mathrm{vol}(\mathbb{S}^{m-p-1})\, \mathrm{vol}(\mathbb{S}^p) \\ &\le \frac{\varepsilon^{m-p-1}}{L^2} \cdot \frac{\varepsilon^{-(m-p-1)}}{m-p-1} \cdot \mathrm{vol}(\mathbb{S}^{m-p-1})\, \mathrm{vol}(\mathbb{S}^p) \quad \text{(by (3.8))} \\ &= \frac{1}{(m-p-1)|\log \varepsilon|^2}\, \mathrm{vol}(\mathbb{S}^{m-p-1})\, \mathrm{vol}(\mathbb{S}^p), \end{aligned}$$

where $L = |\log \varepsilon|$ and $m - p - 1 \ge 1$.

Next, we estimate the denominator of (5.2) from below.

$$
\begin{aligned}
\| \varphi_i \|^2_{L^2(H_1, g_{\varepsilon,p,\overline{k}})} &\geq \int_{\mathbb{S}^p} \int_{C_{\varepsilon,1}} |\chi_i(r)|^2 \cdot \left| v_{\mathbb{S}^p} \right|^2_{g_{\varepsilon,p,\overline{k}}} d\mu_{g_{\mathbb{S}^p}} \, d\mu_{g_{C_{\varepsilon,1}}} \\
&\geq \operatorname{vol}(\mathbb{S}^p) \left\{ \int_{B_1} d\mu_{g_{B_1}} + \int_{-L}^{0} |\chi_i(r)|^2 (\varepsilon \cosh(r))^{m-p-1} \, dr \operatorname{vol}(\mathbb{S}^{m-p-1}) \right\} \\
&\geq \operatorname{vol}(\mathbb{S}^p) \int_{\frac{\pi}{4}}^{\frac{3}{4}\pi} \sin^{m-p-1}(r) dr \operatorname{vol}(\mathbb{S}^{m-p-1}) \\
&\geq \frac{1}{2^{(m-p-1)/2}} \cdot \frac{\pi}{2} \cdot \operatorname{vol}(\mathbb{S}^p) \operatorname{vol}(\mathbb{S}^{m-p-1}).
\end{aligned}
$$

Hence, for φ_1, we obtain an upper bound of the Rayleigh-Ritz quotient:

$$
\frac{\|d\,\varphi_1\|^2_{L^2(H_1, g_{\varepsilon,p,\overline{k}})}}{\|\varphi_1\|^2_{L^2(H_1, g_{\varepsilon,p,\overline{k}})}} \leq \frac{C(m,p)}{|\log \varepsilon|^2}, \tag{5.4}
$$

where $C(m,p)$ is a positive constant depending only on m, p.

In the same way, we can obtain similar upper bounds of the Rayleigh-Ritz quotient for φ_i $(i = 2, 3, \ldots, \overline{k})$:

$$
\frac{\|d\,\varphi_i\|^2_{L^2(H_1, g_{\varepsilon,p,\overline{k}})}}{\|\varphi_i\|^2_{L^2(H_1, g_{\varepsilon,p,\overline{k}})}} \leq \frac{C(m,p)}{|\log \varepsilon|^2}, \tag{5.5}
$$

where $C(m,p)$ is a positive constant depending only on m, p.

Thus, we obtain

$$
\begin{aligned}
\lambda''^{(p)}_k(M, g_{\varepsilon,p,\overline{k}}) &\leq \max_{i=1,\ldots,\overline{k}} \frac{\|d\,\varphi_i\|^2_{L^2(M, g_{\varepsilon,p,\overline{k}})}}{\|\varphi_i\|^2_{L^2(M, g_{\varepsilon,p,\overline{k}})}} \\
&\leq \frac{C(m,p)}{|\log \varepsilon|^2} \longrightarrow 0 \quad (\varepsilon \longrightarrow 0).
\end{aligned} \tag{5.6}
$$

After the normalization of $g_{\varepsilon,p,\overline{k}}$, from Lemma 2.1 (3), Lemma 4.1 and (5.6), it follows that

$$
\begin{aligned}
\lambda''^{(p)}_k(M, \overline{g}_{\varepsilon,p,\overline{k}}) &= \operatorname{vol}(M, g_{\varepsilon,p,\overline{k}})^{\frac{2}{m}} \cdot \lambda^{(p)}_k(M, g_{\varepsilon,p,\overline{k}}) \\
&\leq (A\overline{k} + B)^{\frac{2}{m}} \cdot \max_{i=1,\ldots,\overline{k}} \frac{\|d\,\varphi_i\|^2_{L^2(M, \overline{g}_{\varepsilon,p,\overline{k}})}}{\|\varphi_i\|^2_{L^2(M, \overline{g}_{\varepsilon,p,\overline{k}})}} \\
&\leq (A\overline{k} + B)^{\frac{2}{m}} \frac{C(m,p)}{|\log \varepsilon|^2} \longrightarrow 0 \quad (\varepsilon \longrightarrow 0),
\end{aligned}
$$

where $A, B > 0$ are some constants independent of ε and $\overline{k}$.

This completes all the proofs. □

Remark 5.3. *From the proof of Lemma* 5.1*, we find the following:*

(*i*) *One hyperbolic dumbbell yields one small eigenvalue. Thus, by gluing $\overline{k} = k + b_p(M)$ hyperbolic dumbbells to embedded spheres separately, we obtain another family of Riemannian metrics on M satisfying our desired properties.*

(*ii*) *For the rough Laplacian $\overline{\Delta} = \nabla^*\nabla$ acting on p-forms and tensor fields of any type, the same statement also holds. In fact, in the case of p-forms, for the test p-form in* (5.1)*, since $v_{\mathbb{S}^p}$ is parallel, the same equality as in* (5.3) *also holds:*

$$\nabla \varphi_i = \nabla\big(\chi_i(r)\, v_{\mathbb{S}^p}\big) = \chi'(r) dr \otimes v_{\mathbb{S}^p} \qquad \text{on } H_1.$$

In the case of (a, b)-tensor fields, if we replace $v_{\mathbb{S}^p}$ by

$$\underbrace{\frac{\partial}{\partial r} \otimes \cdots \otimes \frac{\partial}{\partial r}}_{a\text{-times}} \otimes \underbrace{dr \otimes \cdots \otimes dr}_{b\text{-times}}$$

for the test tensor fields like (5.1)*, the same equality still holds.*

Therefore, by the same argument, there exist small $\overline{k} = k + b_p(M)$ eigenvalues of the rough Laplacian $\overline{\Delta}$ acting on p-forms and tensor fields of any type on $(M, \overline{g}_{\varepsilon,p,\overline{k}})$.

6 Remarks and Further Studies

We discuss the future developments of the problem to find a positive lower bound for the first positive eigenvalue of the Hodge-Laplacian on p-forms in terms of geometric quantities (see also [HM24]). It is well-known that the first positive eigenvalue of the Laplacian on functions can be estimated below in terms of the dimension, a lower bound of the Ricci curvature and an upper bound of the diameter ([Gr80], [LY80]). In the case of $1 \leq p \leq m-1$, however, similar estimates do not hold any longer. Typical counterexamples are the Berger spheres collapsing to the complex projective spaces (see [CC90]). In particular, their volumes converge to zero.

Theorem 1.1 (or Theorem 1.3) implies that the first positive eigenvalue of the Hodge-Laplacian acting on p-forms cannot be estimated below in terms of the dimension, the volume and a lower bound of the sectional curvature. From the proof, the diameter for the family of Riemannian metrics diverges to infinity. It is a natural question to ask the case where the diameter is bounded in addition. Colbois and Courtois [CC90, Theorem 0.4] proved the following theorem:

Theorem 6.1 (Colbois and Courtois [CC90])**.** *For given $m \in \mathbb{N}$, $\kappa, v, D > 0$, there exists a positive constant $C(m, \kappa, v, D) > 0$ depending only on m, κ, v and D such* *that any connected oriented closed Riemannian manifold* (M^m, g) *of dimension* m *with* $|K_g| \leq \kappa$*,* $\mathrm{vol}(M, g) \geq v$ *and* $\mathrm{diam}(M, g) \leq D$ *satisfies*

$$\lambda_1^{(p)}(M, g) \geq C(m, \kappa, v, D) > 0$$

for all $p = 0, 1, \ldots, m$*.*

This theorem was proved by contradiction, by means of the $C^{1,\alpha}$-precompactness theorem for $0 < \alpha < 1$ in the Lipschitz topology due to Peters [Pet87, Theorem 4.4]. So, this lower bound is implicit. If the injectivity radius is bounded below away from zero in addition, an explicit lower bound for $\lambda_1^{(p)}(M, g)$ is given by [CT97], [Ma08, Theorem 4.1].

If we weaken the curvature assumption of $|K_g| \leq \kappa$ by $K_g \geq \kappa$, then we do not know whether or not such positive lower bound exists. But, Lott [Lo04, p.918] conjectured the following:

Conjecture 6.2 (Lott [Lo04])**.** *For given* $m \in \mathbb{N}$*,* $\kappa \in \mathbb{R}$ *and* $v, D > 0$*, there would exist a positive constant* $C(m, \kappa, v, D) > 0$ *depending only on* m*,* κ*,* v *and* D *such that any connected oriented closed Riemannian manifold* (M^m, g) *of dimension* m *with* $K_g \geq \kappa$*,* $\mathrm{vol}(M, g) \geq v$ *and* $\mathrm{diam}(M, g) \leq D$ *satisfies*

$$\lambda_1^{(p)}(M, g) \geq C(m, \kappa, v, D) > 0$$

for all $p = 0, 1, \ldots, m$*.*

This conjecture is still open, as far as the authors know. Recently, Honda and Mondino [HM25] obtained a positive lower bound of $\lambda_1^{(1)}(M, g)$ for $p = 1$ under $m \leq 4$, the Ricci curvature $|\mathrm{Ric}_g| \leq \kappa$, $\mathrm{vol}(M, g) \geq v$ and $\mathrm{diam}(M, g) \leq D$. Their lower bound is also implicit with respect to κ, v and D, since their proof is by contradiction, by means of the convergence theory of Riemannian manifolds in dimension 4 combined with the convergence of the eigenvalues for 1-forms [Ho17]. Furthermore, they conjectured the following in the case of $p = 1$:

Conjecture 6.3 (Honda and Mondino [HM25])**.** *For given* $m \in \mathbb{N}$*,* $\kappa \in \mathbb{R}$ *and* $v, D > 0$*, there would exist a positive constant* $C(m, \kappa, v, D) > 0$ *depending only on* m, κ, v *and* D *such that any connected oriented closed Riemannian manifold* (M^m, g) *of dimension* m *with the Ricci curvature* $\mathrm{Ric}_g \geq \kappa$*,* $\mathrm{vol}(M, g) \geq v$ *and* $\mathrm{diam}(M, g) \leq D$ *satisfies*

$$\lambda_1^{(1)}(M, g) \geq C(m, \kappa, v, D) > 0.$$

For given $m \in \mathbb{N}$, $\kappa \in \mathbb{R}$ and $v, D > 0$, we denote by $\mathcal{M}_K$ be the class of all connected oriented closed Riemannian manifolds (M^m, g) of dimension m with $K_g \geq \kappa$, $\mathrm{vol}(M, g) \geq v$ and $\mathrm{diam}(M, g) \leq D$. For Conjecture 6.2, we note that any sequence in $\mathcal{M}_K$ is non-collapsing, and that $\mathcal{M}_K$ has only finite homeomorphism types [GPW90]. Perelman [Per91], [Ka07] proved the topological stability theorem for Alexandrov spaces: Let X be a compact m-dimensional Alexandrov space with the (sectional) curvature bounded below by $\kappa \in \mathbb{R}$. Then, there exists a positive constant $\varepsilon = \varepsilon(X, \kappa) > 0$ such that if any compact m-dimensional Alexandrov space Y with the curvature bounded below by κ satisfies $d_{GH}(X, Y) < \varepsilon$, then X is homeomorphic to Y. Here, d_{GH} denotes the Gromov-Hausdorff distance.

Furthermore, Perelman claimed that the Lipschitz stability theorem held true. Here, the Lipschitz stability theorem means that "homeomorphic" in the topological stability theorem above can be chosen to be "bi-Lipschitz". However, it seems that the paper has not appeared anywhere (see Kapovitch [Ka07, p.104]). If the Lipschitz stability theorem would hold true, then Conjecture 6.2 also holds true, since the class $\mathcal{M}_K$ with the Lipschitz distance is covered with finitely many balls.

In the case of the Ricci curvature bounded below, instead of the sectional curvature, it would be considered that a similar lower bound for $2 \leq p \leq m-2$ does not hold any longer.

Problem 6.4. *Do there exist a closed manifold M of dimension m and a sequence of Riemannian metrics g_i on M with* $\mathrm{Ric}_{g_i} \geq (m-1)\kappa$, $\mathrm{vol}(M, g_i) \geq v > 0$ *and* $\mathrm{diam}(M, g_i) \leq D$ *for uniform constants $\kappa \in \mathbb{R}$, $v, D > 0$ independent of g_i such that for all* $2 \leq p \leq m-2$

$$\lambda_1^{(p)}(M, g_i) \longrightarrow 0 \quad (i \longrightarrow \infty) \ ?$$

We conjecture that the answer to this problem would be positive, however, there exist no such examples ever. To find a lower bound for the first positive eigenvalue of the Hodge-Laplacian acting on p-forms with $2 \leq p \leq m-2$, we may need to control the Weitzenböck curvature tensor. But, it seems to be impossible to control the Weitzenböck curvature tensor for $2 \leq p \leq m-2$, in terms of $\mathrm{Ric}_g \geq -(m-1)\kappa^2$, $\mathrm{vol}(M, g) \geq v$ and $\mathrm{diam}(M, g) \leq D$.

Furthermore, the finiteness theorem for the Ricci curvature version fails. For given $m \in \mathbb{N}$, $\kappa \in \mathbb{R}$ and $v, D > 0$, we denote by $\mathcal{M}_{\mathrm{Ric}}$ the class of all connected oriented closed Riemannian manifolds (M^m, g) of dimension m with $\mathrm{Ric}_g \geq (m-1)\kappa$, $\mathrm{vol}(M, g) \geq v$ and $\mathrm{diam}(M, g) \leq D$. Then, due to the result by Perelman [Per97], there exists infinitely many homeomorphism types in $\mathcal{M}_{\mathrm{Ric}}$. In particular, the p-th Betti number of a closed Riemannian manifold in $\mathcal{M}_{\mathrm{Ric}}$, which is equal to the dimension of the harmonic p-forms, cannot be estimated above in terms of m, κ, v and D. This situation is quite different from the case of the sectional curvature bounded below (cf. the estimate of the total Betti number by Gromov [Gr81]). For further comments and remarks, see the comment by Lott in [Lo18, Remark 4.42].

In contrast, Lott in [Lo18] gave an upper bound of $\lambda_k^{(p)}(M, g)$ under $K_g \geq \kappa$, $\mathrm{diam}(M, g) = D$ and $\mathrm{vol}(M, g) \geq v$. More generally, including collapsing cases, he also gave an upper bound of $\lambda_k^{(p)}(M, g)$ for $0 \leq p \leq n$, in terms of $K_g \geq \kappa$ and the length $\ell > 0$ of an $(n, 1/10)$-strained point with $1 \leq n \leq m$, instead of the assumption $\mathrm{vol}(M, g) \geq v$.

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Colette Anné
Laboratoire de Mathématiques Jean Leray,
Nantes Université, CNRS, Faculté des Sciences,
BP 92208, 44322 Nantes, France
colette.anne@univ-nantes.fr

Junya Takahashi
Research Center for Pure and Applied Mathematics,
Graduate School of Information Sciences,
Tôhoku University,
6–3–09 Aoba, Sendai 980–8579, Japan
t-junya@tohoku.ac.jp